



On some generalizations of an Ostrowski–Grüss type integral inequality[☆]



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ABSTRACT

In this paper, we improve and extend a new Ostrowski–Grüss type integral inequality obtained recently by Feng and Meng (2012) [26]. To be precise, we show that a condition imposed on the inequality can be removed. Furthermore, we provide a two dimensional generalization of this Ostrowski–Grüss type inequality which is sharp.

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1. Introduction

In [1], Ostrowski introduced the following estimate:

Theorem 1.1. *Let $f : I \rightarrow \mathbb{R}$ be a mapping differentiable in the interior of an interval I . Suppose that $a, b \in \text{Int } I$ with $a < b$. If $|f'(t)| \leq M$ for all $t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M \quad \text{for } x \in [a, b]. \quad (1)$$

This inequality can be applied to estimate the deviation of the value of a function from its mean on $[a, b]$. In the literature, the inequality (1) has been extended and generalized in several different ways [2–17]. The concern of this paper is on the inequalities of the Ostrowski–Grüss type [18–25]. This type of inequalities originates from the result of Dragomir and Wang [18]:

Theorem 1.2. *Let I be an open interval and $a, b \in I$ with $a < b$. If $f : I \rightarrow \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma$ for $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma) \quad \text{for all } x \in [a, b]. \quad (2)$$

The bound on the right hand side of (2) was then improved in [19] to $\frac{1}{8} (b-a) (\Gamma - \gamma)$. Furthermore, it was also showed that the constant $\frac{1}{8}$ is sharp in the sense that it can not be replaced by a smaller number.

Recently, Feng and Meng gave a generalization of this type of Ostrowski–Grüss inequality. In [26], Feng and Meng considered an inequality for a division of the interval $[a, b]$:

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$$I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$$

and $m_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \dots, k$ with $m_0 = a$ and $m_{k+1} = b$. They showed that

$$\left| \frac{1}{b-a} \sum_{i=0}^k (m_{i+1} - m_i) f(x_i) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{(b-a)^2} \left[\frac{b^2 - a^2}{2} - \sum_{i=0}^{k-1} m_{i+1} (x_{i+1} - x_i) \right] \right|$$

is bounded by $\frac{1}{8}(b-a)(\Gamma - \gamma)$ if one of the following two conditions holds:

- (H1) there exists $[x_{j-1}, x_j]$, $j = 1, 2, \dots, l_1$ such that $[x_{j-1}, x_j] \cap E^+ \neq \emptyset$, $[x_{j-1}, x_j] \cap E^- \neq \emptyset$ for $j = 1, 2, \dots, l_1$, $E^+ \subset \cup_{j=1}^{l_1} [x_{j-1}, x_j]$ and $m(E^+) \leq \frac{b-a}{2}$,
 (H2) there exists $[x_{n_{j-1}}, x_{n_j}]$, $j = 1, 2, \dots, l_2$ such that $[x_{n_{j-1}}, x_{n_j}] \cap E^+ \neq \emptyset$, $[x_{n_{j-1}}, x_{n_j}] \cap E^- \neq \emptyset$ for $j = 1, 2, \dots, l_2$, $E^+ \subset \cup_{j=1}^{l_2} [x_{n_{j-1}}, x_{n_j}]$ and $m(E^-) \leq \frac{b-a}{2}$.

Here $m(\cdot)$ denotes the measure of a Lebesgue measurable set, E^+ and E^- are the sets on which $p(t, I_k) \geq 0$ and $p(t, I_k) < 0$ respectively, with

$$p(t, I_k) = \begin{cases} t - m_1 - C, & t \in [x_0, x_1), \\ t - m_2 - C, & t \in [x_1, x_2), \\ \vdots & \vdots \\ t - m_{k-1} - C, & t \in [x_{k-2}, x_{k-1}), \\ t - m_k - C, & t \in [x_{k-1}, x_k], \end{cases} \quad (3)$$

where $C = \frac{1}{2}(a+b) - \frac{1}{b-a} \sum_{i=0}^{k-1} m_{i+1} (x_{i+1} - x_i)$.

In the next section, we prove that the conclusion in [26] is still true without the additional conditions (H1) and (H2).

Another concern of this paper is to extend the Ostrowski–Grüss type inequalities to high dimension. A two-dimensional extension was given in [26]. However, it is still not known whether this extension is sharp or not. In this paper, we give another two-dimensional extension of the Ostrowski–Grüss inequality and show that our bound is sharp. This result will also be presented in the next section.

2. Main results

The following lemma constitutes the main ingredient for the proof of the main result in this paper.

Lemma 2.1. Assume that $x_i \in [a, b]$, $i = 1, 2, \dots, k$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a, b]$, and $m_i \in [x_{i-1}, x_i]$, $i = 1, 2, \dots, k$ with $m_0 = a$ and $m_{k+1} = b$. Then we have the following two inequalities

$$\int_a^b \max\{p(t, I_k), 0\} dt \leq \frac{1}{2} [m(E^+)]^2, \quad \int_a^b \min\{p(t, I_k), 0\} dt \geq -\frac{1}{2} [m(E^-)]^2,$$

where $p(t, I_k)$ is given by (3), E^+ and E^- are the sets on which $p(t, I_k) \geq 0$ and $p(t, I_k) < 0$ respectively.

Proof. We first show that

$$\int_a^b \max\{p(t, I_k), 0\} dt \leq \frac{1}{2} [m(E^+)]^2. \quad (4)$$

Denote the points $t_1 < t_2 < \cdots < t_N$ at which $p(t, I_k)$ changes sign. Note that if $p(b, I_k) = 0$, we take $t_N = b$.

We first consider the case that $p(a, I_k) \geq 0$. Since

$$p(a, I_k) = a - m_1 - C \leq b - m_k - C = p(b, I_k), \quad (5)$$

then we get $p(b, I_k) \geq 0$. Notice that $p(t, I_k)$ jumps down at the discontinuous points, we can see that

$$0 \leq p(t, I_k) \leq t - m_1 - C, \quad t \in [a, t_1] \quad (6)$$

and

$$0 \leq p(t, I_k) \leq (t - t_i), \quad t \in [t_i, t_{i+1}] \text{ for even } i \text{ and } t \in [t_N, b]. \quad (7)$$

In particular,

$$b - m_k - C = p(b, I_k) \leq (b - t_N). \quad (8)$$

By using (5)–(8), we get that

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