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On a conditionally stable nonlinear method to approximate some monotone and bounded solutions of a generalized population model

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ABSTRACT

In this work, we design a numerical method to approximate some solutions of a generalization of a nonlinear diffusion–reaction model which appears in the context of population dynamics. The existence of traveling-wave solutions for the equation under consideration is a well-known fact. Some of such solutions are positive, bounded from above, and monotone in both space and time. Motivated by these facts, we propose an explicit, nonlinear, finite-difference methodology to approximate consistently the solutions of the model under investigation. In the linear regime, the method is consistent of first order in time and of second order in space. Under certain, flexible parameter conditions, the method is capable of preserving the positivity, the boundedness, and the spatial and the temporal monotonicity of the traveling-wave solutions. Moreover, we establish analytically and numerically that the nonlinear method is conditionally stable. A computational implementation of our technique shows that the method preserves in practice the mathematical features of interest of the exact solutions considered.

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1. Introduction

The well-known Fisher's equation is one of the simplest diffusive models with nonlinear reaction studied in the mathematical literature. Historically, this equation was investigated simultaneously and independently in 1937 by Fisher [1] and by Kolmogorov et al. [2] in the context of the dynamics of some populations. After the publication of those seminal works, Fisher's equation has been studied extensively from various points of view, and several applications of this model have been proposed to the modeling of epidermal wound healing [3], to the stability of the nonlinear kinetics of nuclear reactors [4] and to the study of diffusive phenomena in population genetics [5] among other applications. In fact, Fisher's equation has been the departing point for extended models in the description of the propagation of forest fires [6], in the study of the thermodynamics of the growth of populations with temporal delays [7] and in the mathematical modeling of population migrations during the Neolithic period [8,9], just to mention some of the many routes of generalization motivated by this equation.

Due in part to the fact that Fisher's equation has found many scientific applications, this model has been investigated extensively in the mathematical and the computational grounds. Thus, it is well-known nowadays that, among its many interesting mathematical features, this equation possesses an infinite number of exact solutions in the form of positive traveling-wave fronts which are bounded from above by a fixed constant [10]. Such solutions are monotone in both space

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and time, and they converge to the nontrivial, stationary solution of the model. Moreover, the existence of this type of solutions for generalizations of Fisher's equation has been analytically established recently using different methodologies [11–15]. Needless to mention that a lot of effort has been devoted also to the general investigation of the existence of solutions of Fisher's equations and their functional properties under different analytical assumptions.

From a computational perspective, Fisher's model has been a source of investigation which has been susceptible to generalization, too. For instance, this equation was studied recently employing the Haar wavelet method [16], through implementations of the variational homotopy perturbation method [17], using split Newton iterative techniques [18], via explicit finite-difference methodologies [19], employing the wavelet-Galerkin method in conjunction with other kinds of discretizations [20], etc. It is important to recall that these techniques have shown good efficiency and a high degree of accuracy when approximating known, exact solutions of Fisher's equation; however, we must remark also that most of these works have been designed without taking into account the properties of symmetry, positivity, boundedness, and spatial and temporal monotonicity which are inherent to some of the traveling-wave solutions of Fisher's model.

In recent works, some effort has been devoted to the design of numerical techniques for generalized Fisher's equations, which guarantee the preservation of the properties of symmetry, positivity, boundedness, and spatial and temporal monotonicity of solutions. Most of these techniques are implicit, semi-linear discretizations of nonlinear problems arising in generalized population models [21], in classical mechanics [22] and in nonlinear fluid dynamics [23,24]. These methods are certainly different in the mathematical nature of the equations that they approximate; however, they share a common methodological feature: The fact that the semi-linear discretization reduces to a linear problem represented by a square, real matrix which, under appropriate conditions, turns out to be an *M-matrix*, that is, a strictly diagonally dominant matrix with positive entries in the main diagonal, and non-positive real numbers in the off-diagonal entries. These matrices are known to be non-singular, and the entries of their inverses are positive, real numbers [25]. Thus, the conditions of positivity and, ultimately, boundedness of the numerical methods are established with little effort. On the other hand, however, this type of discretization makes difficult (it not impossible) to guarantee the spatial and the temporal monotonicity of the approximate solutions, needless to mention the difficulties to provide a nonlinear stability analysis.

In the present work, we attack the problem of approximating consistently the solutions of an extension of Fisher's equation in which the reaction term assumes a generalized logistic law. Our approach will be completely different from previous works in the sense that we will investigate a fully nonlinear, explicit discretization of that equation, as opposed to the semi-linear, implicit techniques used before. As we will see, the nonlinear approach results in many advantages with respect to the semi-linear one. To start with, the computer implementation is easier in view of the fact that each iteration of our method will require an application of Newton's method for the solution of nonlinear equations [26]. In addition to the simplicity of the computational coding of the technique, the preservation of the properties of positivity and boundedness is established analytically in an easier fashion. Moreover, the method preserves the properties of spatial and temporal monotonicity of approximations under the same parameter conditions which guarantee their positivity and boundedness. Our nonlinear technique is skew-symmetry-preserving and conditionally nonlinearly stable. Furthermore, it performs well in practice, as evinced by the numerical simulations obtained through a computational implementation of the method.

Our work is sectioned as follows. In Section 2, we introduce the generalized Fisher's equation that motivates our study. In this stage, we record the analytical expression of a family of traveling-wave solutions for our generalized model, and recall a symmetry property of its solutions under some parametric conditions. Section 3 is devoted to fix the discrete nomenclature, and to introduce the nonlinear finite-difference methodology employed to approximate the solutions of our mathematical model. In Section 4, we establish analytically that our method is capable of preserving the symmetry property derived in Section 2; moreover, we demonstrate that the method preserves the positivity, the boundedness, and the spatial and temporal monotonicity of the numerical approximations, under some relatively relaxed conditions on the computational parameters. Here, we prove that the nonlinear method exhibits conditional stability when it approximates positive and bounded solutions of our generalized Fisher's equation. In the next section, we present some simulations which evince the fact that the method preserves the crucial mathematical characteristics of the solutions, and that it yields good approximations to the exact, traveling-wave solutions recorded previously. We perform a numerical study which confirms the conditionally stable nature of the method, and sheds some light on its convergence. Finally, we close this note with a section of concluding remarks.

2. Mathematical model

Throughout this work, we let I represent a (bounded or unbounded) interval of $\mathbb{R}$, and let $\mathbb{R}^+$ stand for the set of nonnegative, real numbers. Likewise, we suppose that p is a positive integer, and that β and κ are nonnegative, real numbers. Let $u = u(x,t)$ be a real function defined for every $(x,t) \in I \times \mathbb{R}^+$, which is twice differentiable in the interior of its domain. The model investigated in the present work is given by the nonlinear, parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1}$$

where the reaction term assumes the generalized logistic form

$$f(u) = \beta u(1 - u^p). \tag{2}$$

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