Applied Mathematics and Computation 229 (2014) 316–326

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Further results on the reverse order law for the group inverse in rings

Xiaoji Liu [a], Miao Zhang [a], Julio Benítez [b,*]

[a] College of Science, Guangxi University for Nationalities, Nanning 530006, PR China
[b] Instituto de Matemática Multidisciplinar, Universidad Politécnica de Valencia, Valencia 46022, Spain

ARTICLE INFO

Keywords:
Group inverse
Drazin inverse
Moore–Penrose inverse
Reverse order law

ABSTRACT

In this paper, we use the Drazin inverse to derive some new equivalences of the reverse order law for the group inverse in unitary rings. Moreover, if the ring has an involution, we present more equivalences when both involved elements are EP.

1. Introduction

Let $\mathcal{R}$ be a unitary ring whose unity is $\mathbb{1}$. For $a, b \in \mathcal{R}$, the *commutator* of a, b is defined as $[a, b] = ab - ba$. Let $a \in \mathcal{R}$. It can be easily proved that the set of $x \in \mathcal{R}$ satisfying the following conditions

$$axa = a, \quad xax = x, \quad ax = xa \tag{1.1}$$

is either empty or a singleton. When there exists such x, then a is said to be *group invertible* and x is denoted by $a^\#$. A useful fact about group inverses is the following: If a is a group invertible element of a unitary ring $\mathcal{R}$, then $a^\#$ double commutes with a, that is, if $z \in \mathcal{R}$ satisfies $[a, z] = 0$, then $[a^\#, z] = 0$ (see e.g. [6, Lemma 1.4.5]). We shall denote by $\mathcal{R}^\#$ the subset of $\mathcal{R}$ consisting of group invertible elements and by $\mathcal{R}^{-1}$ the set of standard invertible elements. If $a \in \mathcal{R}^\#$, the *spectral idempotent* of a is defined as $a^\pi = \mathbb{1} - aa^\#$. Also, the following result on group inverses will be used. For the proof, the interested reader can consult [15, Proposition 8.22].

Theorem 1.1. *Let a be an element of a unitary ring $\mathcal{R}$. Then a is group invertible if and only if exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$. In this case, one has $a^\# = yax$.*

With each element a of a unitary ring $\mathcal{R}$ we associate two right ideals:

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \quad a^\circ = \{x \in \mathcal{R} : ax = 0\}.$$

Let a, b be elements of a unitary ring $\mathcal{R}$. The element b is a *Drazin inverse* of a if

$$ab = ba, \quad b = ab^2, \quad a^k = a^{k+1}b$$

for some nonnegative integer k. It can be proved (see [7, Theorem 1]) that such b is unique and it is customarily denoted a^d. The least nonnegative integer k for which these equalities hold is the *Drazin index* $i(a)$ of a. In [7, Theorem 4] it was proved that an element $a \in \mathcal{R}$ is Drazin invertible if and only if there are nonnegative integers p, q and $u, v \in \mathcal{R}$ such that $a^{p+1}u = a^p$ and $va^{q+1} = a^q$. The smallest value of p for which $\{u \in \mathcal{R} : a^{p+1}u = a^p\} \neq \emptyset$ is called the *left index* of a, denoted by $l(a)$. In a similar way the *right index* of a is defined, and is denoted by $r(a)$. In a remark following [7, Theorem 4] it was shown that in case that a is Drazin invertible, then $i(a) = l(a) = r(a)$.

* Corresponding author.
E-mail addresses: xiaojiliu72@126.com (X. Liu), zhangmiao198658@163.com (M. Zhang), jbenitez@mat.upv.es (J. Benítez).

0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.12.030

An involution in a ring $\mathcal{R}$ is a map $a \mapsto a^*$ such that for any $a, b \in \mathcal{R}$,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

If the ring $\mathcal{R}$ has an involution, then we can define another class of generalized inverse. An element $a \in \mathcal{R}$ is *Moore–Penrose invertible* if there exists $x \in \mathcal{R}$ such that

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa. \tag{1.2}$$

Such x, when exists, is unique and is denoted by $a^\dagger$.

An element $a \in \mathcal{R}$, where $\mathcal{R}$ is a ring with involution, is said to be *self-adjoint* when $a = a^*$. We say that a is *EP* when a is Moore–Penrose invertible and $aa^\dagger = a^\dagger a$. Observe that when a is EP, then a is group invertible and $a^\dagger = a^\#$, and that any self-adjoint element, Moore–Penrose invertible or group invertible, is EP.

The reverse order law for generalized inverses plays an important role in many areas including singular matrix problem, ill-posed problems, optimization, and statistics (see e.g. [2,8,16–20]. These problems have attracted considerable attention since the middle 1960s and many interesting results have been obtained. Greville [9] proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$, for complex matrices A and B, where $R(\cdot)$ denotes the column space. This result was extended to linear bounded operators on Hilbert spaces in [10]. Another characterization of the reverse order law for the Moore–Penrose inverse is due to Arghiriade [1]: For complex matrices A, B such that AB exists, $(AB)^\dagger = B^\dagger A^\dagger$ if and only if A^*ABB^* is EP. The interested reader can consult [2, Section 4.4] for the proof of the original results of Greville and Arghiriade. Later, the reverse order law for the Moore–Penrose inverse was considered in rings with involution (see [11]). Deng [4] presented some equivalent conditions concerning the reverse order law $(AB)^\# = B^\# A^\#$ for group invertible operators A, B on a Hilbert space. Dinčić and Djordjević [5] gave new equivalences of the reverse order law for the Moore–Penrose inverse for operators on Hilbert spaces. Mosić and Djordjević [13] investigated some necessary and sufficient conditions for the reverse order law for the group inverse in rings. The hybrid reverse order law $(ab)^\# = b^\dagger a^\dagger$ in rings was studied in [14].

In this paper, we give new equivalent conditions of the reverse order law for the group inverse in unitary rings. Later, we state some new results related to the reverse order law for the group inverse when both involved elements are EP.

The word *idempotent* will be reserved for an element p of a unitary ring $\mathcal{R}$ such that $p^2 = p$. Also, we will write $\overline{p} = \mathbb{1} - p$. If in addition $\mathcal{R}$ has an involution, then we will say that an element p is a *projection* when $p = p^2 = p^*$.

2. Preliminary results

If $\mathcal{R}$ is a unitary ring and $p \in \mathcal{R}$ is an idempotent, then every $x \in \mathcal{R}$ has the following matrix representation

$$x = \begin{bmatrix} pxp & px\overline{p} \\ \overline{p}xp & \overline{p}x\overline{p} \end{bmatrix}_p.$$

If, in addition, $\mathcal{R}$ has an involution and p is a projection, then the above matrix representation preserves this involution. More precisely, if $x \in \mathcal{R}$ is represented as $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_p$, then $x^* = \begin{bmatrix} x_1^* & x_3^* \\ x_2^* & x_4^* \end{bmatrix}_p$.

If a is an element of a ring $\mathcal{R}$, then

$$a \in \mathcal{R}^\# \iff \text{there is an idempotent } p \in \mathcal{R} \text{ such that } a + p \in \mathcal{R}^{-1} \text{ and } ap = pa = 0. \tag{2.1}$$

Such a p, when it exists, is unique (see [15, Proposition 8.24]). This unique idempotent p is the spectral idempotent of a – recall that the spectral idempotent of a is customarily written by a^π and $a^\pi = \mathbb{1} - aa^\#$ holds. Hence, if $a \in \mathcal{R}^\#$, we can represent

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{aa^\#}, \quad a^\# = \begin{bmatrix} a^\# & 0 \\ 0 & 0 \end{bmatrix}_{aa^\#}, \quad a^\pi = \begin{bmatrix} 0 & 0 \\ 0 & a^\pi \end{bmatrix}_{aa^\#}. \tag{2.2}$$

When (2.1) is applied to a group invertible matrix $A \in \mathbb{C}^{n \times n}$, by writing the idempotent A^π as $U(0 \oplus I_k)U^{-1}$, where $U \in \mathbb{C}^{n \times n}$ is nonsingular and I_k denotes the identity matrix of order k (see e.g. [21, Theorem 5.1]), one easily gets the existence of a nonsingular matrix $B \in \mathbb{C}^{(n-k)\times(n-k)}$ such that $A = U(B \oplus 0)U^{-1}$. Obviously, we have also $A^\# = U(B^{-1} \oplus 0)U^{-1}$.

Lemma 2.1. *Let $\mathcal{R}$ be a unitary ring and $m \in \mathcal{R}$. If $p \in \mathcal{R}$ is an idempotent,*

$$m = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}_p, \tag{2.3}$$

then

(i) *If m is group invertible and c is Drazin invertible, then a, c are group invertible and $a^\pi bc^\pi = 0$.*
(ii) *If a, c are group invertible and $a^\pi bc^\pi = 0$, then m is group invertible and*