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Analytical and numerical study for an integro-differential nonlinear Volterra equation

H. Guebbai *, M.Z. Aissaoui, I. Debbar, B. Khalla

Laboratoire de Mathématiques Appliquées et de Modélisation, Faculté de Mathématiques et de l'Informatique et des Sciences de la Matiére, Université 8 Mai 1945 Guelma, B.P. 401 Guelma 24000, Algeria

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A B S T R A C T

In this work we study the solution's existence and uniqueness for an integro-differential nonlinear Volterra equation and then we approximate the solution of this equation by using Nyström method.

1. Introduction

The purpose of the present paper is to study an integro-differential nonlinear Volterra equation. This type of equation is very important in physics and mathematics. The non-differential case was much studied compared to the differential case (see [1–3]). The integro-differential equations which have been studied are the case that the derivative of the unknown is outside the integral. The equation that we study in this paper is the following:

$$u(t) = \int_a^t K(t,s,u(s),u'(s))ds + f(t); \quad \forall t \in [a,b], \tag{1}$$

where, $f \in C^1([a,b])$ and u is unknown, to be found in the same space. This equation is similar to those studied in [4].

Firstly, we study this equation in analytical sense i.e. we show the existence and the uniqueness of the solution, using ideas based on the Schauder's fixed point theorem and similar to those used in [2,3].

Secondly, we study this equation in numerical sense: We use a Nyström method (see [1]) to approximate the solution. Finally, we give numerical examples.

2. The equation

Let K be a function defined by,

$$K : [a,b]^2 \times \mathbb{R}^2 \to \mathbb{R}, (t,s,x,y) \mapsto K(t,s,x,y).$$

We suppose that K satisfy the following hypotheses:

* Corresponding author.
E-mail addresses: guebaihamza@yahoo.fr (H. Guebbai), aissaouizine@gmail.com (M.Z. Aissaoui), ilyasdebbar@yahoo.fr (I. Debbar), basema.khalla@yahoo.fr (B. Khalla).

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$$(H1)\left\|\begin{array}{ll}(1) & \frac{\partial K}{\partial t}\in C([a,b]^2\times\mathbb{R}^2),\\ (2) & \exists M\in\mathbb{R}_+^*,\ \forall t,s\in[a,b],\ \forall x,y\in\mathbb{R},\\ & \max(|K(t,s,x,y)|,|\frac{\partial K}{\partial t}(t,s,x,y)|)\leqslant M.\end{array}\right.$$

From this function, and for $f\in C^1(a,b)$, we define the functional Φ_f by:

$$\forall\xi\in C^1(a,b),\forall t\in[a,b],\quad \Phi_f(\xi)(t):=\int_a^t K(t,s,\xi(s),\xi'(s))ds+f(t).$$

Proposition 1. *For all $f\in C^1(a,b),\Phi_f$ is continuous from $C^1([a,b])$ to it self.*

Proof. Let $\xi\in C^1(a,b)$, it is clear that $\Phi_f(\xi)$ is continuous on $[a,b]$. For all $t\in]a,b[$, (see [3])

$$\Phi_f(\xi)'(t):=K(t,t,\xi(t),\xi'(t))+\int_a^t\frac{\partial K}{\partial t}(t,s,\xi(s),\xi'(s))ds+f'(t),$$

which is continuous over $]a,b[$ and bounded by $(b-a+1)M+\|f\|_{C^1(a,b)}$. Then, $\Phi_f(\xi)\in C^1(a,b)$.

Let $\{\xi_n\}_{n\in\mathbb{N}}$ a sequence in $C^1(a,b)$, which converges to $\xi\in C^1(a,b)$. We have, $\forall t\in[a,b]$

$$\begin{aligned}\lim_{n\to+\infty}\Phi_f(\xi_n)(t)&=\lim_{n\to+\infty}\left(\int_a^t K(t,s,\xi_n(s),\xi_n'(s))ds+f(t)\right)=\int_a^t\lim_{n\to+\infty}K(t,s,\xi_n(s),\xi_n'(s))ds+f(t)\\&=\int_a^t K(t,s,\lim_{n\to+\infty}\xi_n(s),\lim_{n\to+\infty}\xi_n'(s))ds+f(t)=\Phi_f(\xi)(t).\quad\square\end{aligned}$$

Consider the following nonlinear integro-differential equation:

$$u(t)=\int_a^t K(t,s,u(s),u'(s))ds+f(t);\quad\forall t\in[a,b],\tag{2}$$

where, $f\in C^1([a,b])$ and u is unknown, to be found in the same space. This equation contains more information about the solution u: If both sides of the equation are derived, we obtain:

$$u'(t)=K(t,t,u(t),u'(t))+\int_a^t\frac{\partial K}{\partial t}(t,s,u(s),u'(s))ds+f'(t);\quad\forall t\in[a,b],\tag{3}$$

The purpose of this paper, is to find conditions to ensure the existence and uniqueness of the solution. Then build a numerical technique to approach it.

3. Analytical study

3.1. Existence

Theorem 2. *Eq. (2) has a solution in $C^1(a,b)$.*

Proof. We define the following set:

$$F:=\{\xi\in C(a,b):\ \xi(a)=f(a),\ \forall t\in[a,b],\ |\xi(t)-f(t)|\leqslant M(b-a),|\xi'(t)-f'(t)|\leqslant M(b-a+1)\}.$$

It is clear that the set F is closed and convex. For all $\xi\in F$ and all $t\in[a,b]$

$$\begin{aligned}&\Phi(\xi)(a)=f(a),\\&|\Phi_f(\xi)(t)-f(t)|=\left|\int_a^t K(t,s,\xi(s),\xi'(s))ds\right|,\\&\leqslant M(b-a),\\&|\Phi(\xi)'(t)-f'(t)|=\left|K(t,t,\xi(t),\xi'(t))+\int_a^t\frac{\partial K}{\partial t}(t,s,\xi(s),\xi'(s))ds\right|,\\&\leqslant M(b-a+1).\end{aligned}$$

Then, $\Phi_f(F)\subset F$. But, Φ_f is continuous on $C^1(a,b)$ to itself and for all $t,t'\in[a,b]$