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L_2 error estimates of collocation methods for solving certain singular integral equations

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ABSTRACT

In this paper, we propose some special collocation schemes for solving Hilbert type singular and hypersingular integral equations on a circle, based on the superconvergence of quadrature rules for evaluating the corresponding singular integrals. With the aid of spectral analysis, the optimal and suboptimal L_2 error estimates are established in a unified framework. At last, several numerical examples are provided to confirm the theoretical analysis.

1. Introduction

Consider the singular integral equation

$$\mathcal{H}\varphi(s) = \int_0^{2\pi} \kappa(t-s)\varphi(t)\,dt = g(s), \quad s \in (0, 2\pi), \tag{1}$$

where $g(s)$ is a given function, $\varphi(t)$ is unknown density function, $\kappa(t)$ is a singular integral kernel, may be $\frac{1}{2\pi}\cot\frac{t}{2}$ or $\frac{1}{4\pi\sin^2(t/2)}$ and all of them are 2π-periodic. If $\kappa(t) = \frac{1}{2\pi}\cot\frac{t}{2}$, (1) is often called as Hilbert singular integral equation, and the integral in the left side must be understood in Cauchy principal value sense. If $\kappa(t) = \frac{1}{4\pi\sin^2(t/2)}$, (1) becomes a hypersingular integral equation, and the integral in the left side must be understood in Hadamard finite part sense. Such equations are frequently encountered in physical and engineering applications, such as in fracture mechanics, elasticity problems, aerodynamics as well as electromagnetic scattering [6,7].

Numerous works have been devoted in developing efficient numerical methods for the solution of such equations. For example, Chandler [3] studied a Cauchy singular integral equation on a smooth closed curve by using a simple midpoint collocation method, and some convergence results, but not optimal, were obtained. By using spectral analysis, Yan [12,13] provided the convergence analysis of the midpoint collocation method for solving boundary integral equations with logarithmic kernel on a closed boundary. Based on trigonometric interpolation, a fully discrete method was suggested for solving the hypersingular boundary integral equation arising from scattering problem by Kress [6], where an exponential convergence rate was proved for analytic boundaries and boundary data. Mülthei and Schenider [10] proposed a fully discrete collocation scheme on graded mesh for solving Hilbert singular integral equations, and some convergence results in sup-norms and weighted L_2-norms were obtained. Schneider investigated the stability of the pseudo-inverse of discretized Hilbert transform in [11]. In [4], a unified framework for various collocation methods of numerical solutions of Hilbert singular integral equations were established. A fast Fourier–Galerkin method for solving a class of singular boundary integral equations was developed in [2] by compressing the dense Galerkin matrix to a sparse one. In [5], the authors studied a collocation scheme based

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on the superconvergence result of midpoint rule for solving hypersingular integral equation on a circle, where the optimal error estimate was established by analyzing the properties of the scheme's coefficient matrix.

In this paper, we extend the spectral analysis proposed in [12] to analyze the collocation method based on the superconvergence results of midpoint and trapezoidal rules for solving (1). Superconvergence analysis of Newton–Cotes rules for singular integrals has been investigated in depth for the past few years [9,15,16,19]. However, up to now, most of these are restricted on the analysis of the quadrature rules for singular integrals. Only a few error estimates have been obtained for the solution of singluar integral equations, such as in [5,17], where the optimal convergence rates in maximum norm were obtained, but the analytic process rely on the properties of the resulted matrix too much. Through the spectral analysis, we investigate the error estimates in a unified framework, where the eigenvalues of the considered schemes can be expressed explicitly, and thus the spectral norm of the resulted coefficient matrices can be bounded. Combined with the superconvergence results of the quadrature rules, the optimal and suboptimal discrete L_2 error estimates are established.

The rest of this paper is organized as follows. In Section 2, we propose the collocation schemes for the solution of (1) in a unified way and then the general result is given. The error estimates are given for Hilbert singular integral equation and hypersingular integral equation in Sections 3 and 4, respectively. Then, we provide the proof of superconvergence results of midpoint rule for evaluating Hilbert singular integrals in Section 5. Finally, several numerical examples are presented to confirm our theoretical analysis.

2. General result

For simplicity of exposition, we confine ourselves to the case where $0 = t_0 < t_1 < t_2 < \cdots < t_n = 2\pi$ is a uniform mesh of $[0, 2\pi]$ with the mesh size $h = 2\pi/n$. Denote the interpolation of φ by

$$\Pi_h\varphi(t) = \sum_{j=0}^{n-1}\varphi(p_j)\chi_j(t), \tag{2}$$

where $p_j \in [t_j, t_{j+1})$ denote the interpolation points and χ_j the corresponding basis function. For example, if p_j is chosen as $\hat{t}_j = \frac{t_j+t_{j+1}}{2}$ and

$$\chi_j(t) = \begin{cases} 1, & \text{on } [t_j, t_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$\Pi_h\varphi$ is the piecewise constant interpolation of φ; if p_j is chosen as t_j and

$$\chi_j(t) = \begin{cases} \frac{t-t_{j-1}}{h}, & \text{on } [t_{j-1}, t_j], \\ \frac{t_{j+1}-t}{h}, & \text{on } [t_j, t_{j+1}], \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

with $t_{-1} + 2\pi = t_{n-1}$, then $\Pi_h\varphi$ is the piecewise linear interpolation of φ. Substituting $\Pi_h\varphi$ with φ defined in (1) yields the quadrature rule of $\mathcal{H}\varphi$

$$\mathcal{Q}_h\varphi(s) = \int_0^{2\pi}\kappa(t-s)\Pi_h\varphi(t)\,dt = \sum_{j=0}^{n-1}\varpi_j(s)\varphi(p_j) = \mathcal{H}\varphi(s) + \mathcal{E}\varphi(s), \tag{5}$$

where $\mathcal{E}\varphi$ denotes the error functional and

$$\varpi_j(s) = \int_0^{2\pi}\kappa(t-s)\chi_j(t)\,dt$$

is the quadrature coefficient. Applying the rule (5) to approximate the integral in (1), we get

$$\sum_{j=0}^{n-1}\varpi_j(s)\varphi(p_j) \approx g(s). \tag{6}$$

Then, collocating this equation at the points $s_i \in (t_i, t_{i+1})$ yields

$$\sum_{j=0}^{n-1}\varpi_j(s_i)\varphi_j = g(s_i), \quad i = 0, 1, \ldots, n-1, \tag{7}$$

where φ_j denote the approximate value of φ at the point p_j. For the sake of analysis, we rewrite it into the matrix form

$$\mathbb{A}\boldsymbol{\varphi}^a = \mathbf{g}^e, \tag{8}$$

where