Applied Mathematics and Computation 229 (2014) 467–479

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical approximation of an interface problem for fractional in time diffusion equation ☆

Aleksandra Delić *, Boško S. Jovanović

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia

ARTICLE INFO

Keywords:
Fractional in time diffusion equation
Initial-boundary value problem
Interface problem
Finite differences
Error bound

ABSTRACT

An initial-boundary value problem for fractional in time diffusion equation with interface is considered. Its well-posedness in the corresponding Sobolev spaces is proved. Some finite difference schemes approximating the problem are proposed and their stability and convergence are investigated.

1. Introduction

In recent years, fractional partial differential equations have become increasingly popular due to their frequent appearance in various applications in many fields of science and engineering including viscoelasticity, fluid mechanics, electrical networks, electromagnetic theory and probability (see [1–5]).

Because of the integral in the definition of the fractional order derivatives, it is apparent that these derivatives are non-local operators. This explains one of their most significant uses in applications: non-integer derivatives possess a memory effect which it shares with several materials such as viscoelastic materials or polymers. On the other side, this feature of the fractional derivatives makes the design of accurate and fast numerical methods difficult.

The time fractional diffusion equation describes anomalous sub-diffusion corresponding to $0 < \alpha < 1$. It has been investigated by many authors. For example, Alikhanov in [6] has shown that the method of energy inequalities can be applied to obtain a priori estimates for boundary value problems for the sub-diffusion equation in differential and difference settings exactly as in the classical case ($\alpha = 1$). Lin and Xu [7] examined a finite difference/Legendre spectral method to solve the initial-boundary value time-fractional diffusion problem on a finite domain and they obtained estimates of $(2-\alpha)$-order convergence in time and exponential convergence in space. Sun and Wu [8] derived a fully discrete difference scheme for the time fractional diffusion equation. They have proved the stability and convergence by the energy method.

In this article, we consider an initial-boundary value problem for fractional in time diffusion equation with interface. Its well posedness in the corresponding Sobolev spaces is proved. Some finite difference schemes approximating the problem are proposed and their stability and convergence are investigated. Analogous problem for integer order diffusion equation is considered in [9–11].

The layout of the paper is as follows. In Section 2 we introduce Riemann–Liouville fractional derivative. In Section 3 we define some function spaces containing functions with fractional derivatives and expose some their properties. In Section 4 we briefly expose the properties of initial-boundary-value problem for fractional in time diffusion equation with interface and prove existence and uniqueness of its weak solution. In Section 5 we introduce meshes, finite-difference operators

☆ This work was supported by Ministry of Education, Science and Technological Development of Republic of Serbia through project No. 174015.

* Corresponding author.
E-mail addresses: adelic@matf.bg.ac.rs (A. Delić), bosko@matf.bg.ac.rs (B.S. Jovanović).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.12.060

and discrete Sobolev-like norms and also define an finite difference scheme approximating considered problem for fractional in time diffusion equation. Section 6 is devoted to the error analysis of proposed finite difference schemes.

2. Fractional derivatives

Let u be a function defined on the $\mathbb{R}, \alpha > 0$ and n be the smallest integer greater than α. Then the left Riemann–Liouville fractional derivative of order α is defined to be [1]

$$D_+^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^t \frac{u(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{1}$$

where $\Gamma(\cdot)$ denotes Gamma function. The right Riemann–Liouville fractional derivative is defined analogously:

$$D_-^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_t^\infty \frac{u(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau. \tag{2}$$

If $\text{supp}(u) \subset (a,b)$, then

$$D_+^\alpha u(t) = D_{a+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{u(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau$$

and

$$D_-^\alpha u(t) = D_{b-}^\alpha u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_t^b \frac{u(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau.$$

3. Some function spaces

First, we introduce some notations and define some function spaces, norms and inner products that are used thereafter. Let Ω be an open domain in $\mathbb{R}^n$. As usual, by $C^k(\Omega)$ and $C^k(\overline{\Omega})$ we denote the spaces of k-fold differentiable functions. By $\dot{C}^\infty(\Omega) = C_0^\infty(\Omega)$ we denote the space of infinitely differentiable functions with compact support in Ω. The space of measurable functions whose square is Lebesgue integrable in Ω is denoted by $L^2(\Omega)$. The inner product and norm of $L^2(\Omega)$ are defined by

$$(u,v)_\Omega = (u,v)_{L^2(\Omega)} = \int_\Omega u v \, d\Omega, \quad \|u\|_\Omega = \|u\|_{L^2(\Omega)} = (u,u)_\Omega^{1/2}.$$

We also use $H^\alpha(\Omega)$ and $\dot{H}^\alpha(\Omega) = H_0^\alpha(\Omega)$ to denote the usual Sobolev spaces [12], whose norms are denoted by $\|u\|_{H^\alpha(\Omega)}$.

For $\alpha > 0$ let us set

$$|u|_{H_+^\alpha(a,b)} = \|D_{a+}^\alpha u\|_{L^2(a,b)}, \quad |u|_{H_-^\alpha(a,b)} = \|D_{b-}^\alpha u\|_{L^2(a,b)},$$

and

$$\|u\|_{H_\pm^\alpha(a,b)} = \left(\|u\|_{L^2(a,b)}^2 + |u|_{H_\pm^\alpha(a,b)}^2 \right)^{1/2}.$$

Then we define the spaces $H_\pm^\alpha(a,b)$ and $\dot{H}_\pm^\alpha(a,b)$ as the closure of $C^\infty(a,b)$ and $\dot{C}^\infty(a,b)$, respectively, with respect to the norm $\|\cdot\|_{H_\pm^\alpha(a,b)}$. Because for $\alpha = n \in \mathbb{N}$, fractional derivative reduces to standard integer order derivative, we have $H_\pm^n(a,b) = H^n(a,b)$.

Lemma 1 (*see [13,14]*). *Let* $0 < \alpha < 1,\ u \in H_+^\alpha(a,b)$ *and* $v \in H_-^\alpha(a,b)$. *Then*

$$(D_{a+}^\alpha u,\ v)_{L^2(a,b)} = (u,\ D_{b-}^\alpha v)_{L^2(a,b)}.$$

Lemma 2 (*see [15]*). *Let* $\alpha > 0$, $u \in \dot{C}^\infty(\mathbb{R})$ *and* $\text{supp}\, u \subset (a,b)$. *Then*

$$(D_{a+}^\alpha u, D_{b-}^\alpha u)_{L^2(a,b)} = \cos \pi\alpha \|D_{a+}^\alpha u\|_{L^2(a,+\infty)}^2.$$

For $\alpha > 0,\ \alpha \neq n + 1/2,\ n \in \mathbb{N}$, we set

$$|u|_{H_c^\alpha(a,b)} = |(D_{a+}^\alpha u, D_{b-}^\alpha u)_{L^2(a,b)}|^{1/2}, \quad \|u\|_{H_c^\alpha(a,b)} = \left(\|u\|_{L^2(a,b)}^2 + |u|_{H_c^\alpha(a,b)}^2 \right)^{1/2}$$

and define the space $\dot{H}_c^\alpha(a,b)$ as the closure of $\dot{C}^\infty(a,b)$ with respect to the norm $\|\cdot\|_{H_c^\alpha(a,b)}$.