Applied Mathematics and Computation 227 (2014) 50–65

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Global dynamics of quadratic second order difference equation in the first quadrant

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ARTICLE INFO

Keywords:
Attractivity
Basins
Difference equation
Invariant
Period-two solutions
Stable manifold
Unstable manifold

ABSTRACT

We investigate the global behavior of a quadratic second order difference equation

$$x_{n+1} = Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F, \quad n = 0, 1, \ldots$$

with non-negative parameters and initial conditions. We find the global behavior for all ranges of parameters and determine the basins of attraction of all equilibrium points.

1. Introduction

Consider the following difference equation

$$x_{n+1} = Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F, \quad n = 0, 1, \ldots \tag{1}$$

with non-negative parameters and initial conditions. We assume that $A + B + C > 0$ in order to avoid trivial linear case from our consideration and also $A + B + C > 0$ and $A + B + D > 0$, $B + C + E > 0$ in order to avoid the well-known cases of first order quadratic difference equations. Polynomial difference equations and corresponding maps have been studied in both real and complex domain and many results have been obtained, see [3–7,10]. Most known case of quadratic polynomial equations is Hénon equation

$$x_{n+1} = 1 + bx_{n-1} - ax_n^2, \quad n = 0, 1, \ldots \tag{2}$$

which was considered by many authors, see [3–5,10] and references therein. Eq. (2) is a special case of Eq. (1). In this paper we restrict our attention to non-negative initial conditions and non-negative parameters which will make our results more special but also more precise and applicable. First papers on quadratic polynomial difference equations with non-negative initial conditions and non-negative parameters are [1,2], where the special case

$$x_{n+1} = Bx_nx_{n-1} + Ex_{n-1} + F, \quad n = 0, 1, \ldots \tag{3}$$

of Eq. (1) was considered and results on global dynamics have been obtained. The obtained results in [1,2] provided the parts of the basins of attraction of different equilibrium points and periodic solutions. In this paper we give the precise description of all basins of attraction of all attractors of Eq. (1) as well as the point at infinity. In particular, we find the regions of the plane of initial conditions for which all solutions are bounded as well as the escape regions, that is the basins of attraction

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.10.048

of ∞. Such sets have complicated structure in complex dynamics, see [3,6,7]. Our results are based on number of theorems which hold for monotone difference equations and corresponding monotone maps, which will be described in the next section. An interesting consequence of our results is the necessary and sufficient condition for the existence of period-two solutions. For example, if $D = E = F = 0$ that is Eq. (1) contains only quadratic terms, the necessary and sufficient condition for the existence of the period-two solution is $C > 3A + B$, in which case the period-two solution is a saddle point which stable manifold serves as the separatrix between two basins of attraction of a locally stable equilibrium point and the point at ∞.

The paper is organized as follows. The Section 2 presents the preliminary results from the theory of monotone maps. The Section 3 briefly gives the local stability analysis of all equilibrium points for all values of parameters. The Section 4 contains stability analysis of period-two solution which plays major role in our results. The Section 5 gives global results in all regions of parameters. The Section 6 provides two illustrative examples of global dynamics of some special cases of Eq. (1).

2. Preliminaries

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{4}$$

where f is a continuous and increasing function in both variables. There are several global attractivity results for Eq. (4) which give the sufficient conditions for all solutions to approach a unique equilibrium. These results were used efficiently in monograph [14] to study the global behavior of solutions of second order linear fractional difference equation. Here we list some of these results that will be needed in this paper.

The first result was obtained in [14] and it was extended to the case of higher order difference equations and systems in [12,15,16,18].

Theorem 1. *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(x, y)$ is non-decreasing in each of its arguments;*
(b) *Eq. (4) has a unique equilibrium $\bar{x} \in [a, b]$.*

Then every solution of Eq. (4) converges to $\bar{x}$.

The following result has been obtained in [1].

Theorem 2. *Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of Eq. (4) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

(i) *Eventually they are both monotonically increasing.*
(ii) *Eventually they are both monotonically decreasing.*
(iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

As a consequence of Theorem (2) every bounded solution of Eq. (1) approaches either an equilibrium solution or period-two solution or the finite point at the boundary and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of Eq. (1) is the problem of determining the basins of attraction of three different types of attractors: the equilibrium solutions, period-two solution(s) and the point(s) at infinity. The following two results can be proved by using the techniques of proof of Theorem 11 in [8].

Theorem 3. *Consider Eq. (4) where f is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2),$$

that is $x_1 \leqslant x_2,\ y_1 \leqslant y_2$. Assume that E_1 is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$, with the neighborhoods where f is strictly increasing and E_2 is a local attractor.

Then the basin of attraction $\mathcal{B}(E_2)$ of E_2 is the region above the global stable manifolds $\mathcal{W}^s(E_1)$. More precisely

$$\mathcal{B}(E_2) = \{(x, y) : \exists y_l : \ y_l < y, \quad (x, y_l) \in \mathcal{W}^s(E_1)\}.$$

The basin of attraction $\mathcal{B}(E_1) = \mathcal{W}^s(E_1)$ is exactly the global stable manifold of E_1. If there exists a period-two solution, then the end points of the global stable manifold are exactly the period two solution.