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Three dimensional simulation of radiation induced convection

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ABSTRACT

Convection induced by the selective absorption of radiation is investigated, where the internal heat source is concentration dependent. Regions of very large subcritical instabilities, i.e. where agreement between the linear instability thresholds and nonlinear stability thresholds is poor, are studied by solving for the full three-dimensional system. The results indicate that linear theory is very accurate in predicting the onset of convective motion, and thus, regions of stability.

1. Introduction

Convection due to the selective absorption of radiation has been shown experimentally by Krishnamurti [1] for a viscous fluid. The convection mechanism is essentially a penetrative one (which occurs when buoyancy driven motion penetrates into stably stratified layers) effectively modeled via an internal heat source. Due to the wide range of industrial and geophysical applications, extensive literature has been recently produced on penetrative convection, see e.g. Carr [2], Carr and de Putter [3], Carr and Straughan [4], Hill [5], Aziz and Aziz [6], Cloete and Smit [7], Sharma [8], Hasan et al. [9], Singh et al. [10], Mahajan and Monika Arora [11], Pal et al. [12] and Srivastava et al. [13].

The experiment performed by Krishnamurti [1] involves the use of a layer of water containing thymol blue. Thymol blue is predominantly orange substance which becomes blue (the conjugated form) in a high pH. Introducing a positive electrode along the bottom layer of the system produces hydroxyl ions, which precipitates a high pH. Effectively there is diffusion, due to the pH differential, of the blue coloring into the orange from the bottom to top layer. The key part of the experiment is the introduction of a sodium lamp which emits orange radiation that is only significantly absorbed by the blue form of the fluid. This has the effect of causing the less dense blue fluid to rise, creating a convective motion.

Krishnamurti [1] developed a model to describe this experiment, where the internal heat source was assumed to be linearly proportional to the concentration field of the conjugated form of the thymol blue. Straughan [14] further developed the model by studying its linear instability and nonlinear stability. Straughan [14] also employed realistic boundary conditions, appropriate to fixed surfaces, which more accurately reflect the physical experiments. The results of [14] lend much credence to use of the model introduced in [1] for radiation induced convection. Wicks and Hill [15] studied the linear and nonlinear stability analyses of double-diffusive convection in a fluid layer, with a concentration-based internal heat source present. Chang [16] considers this system in a fluid overlying a porous medium. Olali [17] presented a linear instability analysis for the inception of double-diffusive convection with a concentration based internal heat source. The system encompasses a layer of fluid which lies above a porous layer saturated with the same fluid. Hill [5,18–22] explores this system through the use of porous materials.

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In this paper we investigate the onset of convection in Krishnamurti's model [1] to isolate regions of subcritical instability (where the linear instability and nonlinear stability thresholds do not agree), and then explore these regions for the full three dimensional system (using a velocity–vorticity formulation in order to employ second order finite difference schemes). We use both implicit and explicit schemes to enforce the free divergence equation. The size of the physical space explored is evaluated according to the normal modes representation, where periodic boundary conditions for velocity, temperature, and concentration in the x, y dimensions are employed. Standard indicial notation is employed throughout, where $(x_1, x_2, x_3) = (x, y, z)$.

2. Governing equations

Let us consider a fluid layer bounded by two horizontal parallel planes. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (0, d)$ and $Oxyz$ be a Cartesian frame of reference with unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, respectively.

The momentum and conservation of mass equations for a linear viscous fluid [1,14] are

$$v_{i,t} + v_j v_{i,j} = -\frac{1}{\rho_0} p_{,i} + \nu \Delta v_i - \frac{g}{\rho_0} k_i \rho, \tag{2.1}$$

$$v_{i,i} = 0, \tag{2.2}$$

where Δ is the Laplacian, v_i and p, the velocity and pressure, $\mathbf{k} = (0, 0, 1)$, g is acceleration due to gravity, and ν the kinematic viscosity. Denoting T to be the temperature, we take the density ρ to be of the form

$$\rho(T) = \rho_0(1 - \alpha(T - T_0)),$$

where ρ_0 and T_0 are reference density and temperature values, respectively, and α is the coefficient of thermal expansion. The heat equation governing the temperature field, is defined as

$$T_{,t} + v_i T_{,i} = \kappa \Delta T + \beta C, \tag{2.3}$$

where C is the concentration of the conjugated form of the thymol blue, β is a constant of proportionality and κ is the thermal diffusivity. The concentration of the conjugated form of thymol blue follows the convection–diffusion equation

$$C_{,t} + v_i C_{,i} = \kappa_C \Delta C, \tag{2.4}$$

where κ_C is the diffusivity of the conjugated form. The upper and lower planes are held fixed at temperatures T_U and T_L, respectively, and by applying fixed voltages to the conducting upper and lower boundaries, we can maintain concentrations 0 and c_L, respectively, as is shown in [1].

The basic steady-state solution in whose stability we are interested, is where there is no fluid flow (i.e. $\overline{v}_i \equiv 0$). The steady-state concentration $\overline{c}(z)$ can be derived from (2.4), such that

$$\overline{c}(z) = \frac{c_L}{d}(d - z). \tag{2.5}$$

Similarly, applying the fixed temperatures and integrating, the steady-state temperature field can be derived from (2.3) as

$$\overline{T}(z) = T_L + \left(-\frac{H\Delta T}{d} + \frac{d\beta c_L}{3\kappa}\right)z - \frac{\beta c_L}{2\kappa}z^2 + \frac{\beta c_L}{6d\kappa}z^3, \tag{2.6}$$

where $\Delta T = T_L - T_U$ and $H = \mathrm{sgn}(\Delta T)$. The steady state pressure $\overline{p}$ may be calculated from (2.1) (it plays no role in the ensuing analysis).

We employ the same non-dimensionalisation as in [14] except, that we when are heating from above ($T_U > T_L$) in the layer, we replace R^2 by $-R^2$ and γ with $-\gamma$. In this way we derive the fully nonlinear perturbation equations,

$$Pr^{-1}(u_{i,t} + u_j u_{i,j}) = -\pi_{,i} + \Delta u_i + R\theta k_i, \tag{2.7}$$

$$u_{i,i} = 0, \tag{2.8}$$

$$\theta_{,t} + u_i \theta_{,i} = \gamma R \phi + HRF(z)w + \Delta\theta, \tag{2.9}$$

$$\phi_{,t} + u_i \phi_{,i} = w + \eta \Delta\phi. \tag{2.10}$$

In these equations, (u_i, π, θ, ϕ) are perturbations to $\overline{v}_i, \overline{p}, \overline{T}, \overline{C}$, i.e. $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{u}$, $p = \overline{p} + \pi$, $T = \overline{T} + \theta$, $C = \overline{C} + \phi$, $w = u_3$, and the equations hold on $\{z \in (0, 1)\} \times \{(x, y) \in \mathbb{R}^2\}$. Furthermore, $Pr = \nu/\kappa$ is the Prandtl number, and R^2 the Rayleigh number defined by

$$R^2 = \frac{\alpha g}{\nu\kappa}|\Delta T| d^3. \tag{2.11}$$