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Positive solutions for singular semipositone boundary value problems on infinite intervals

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ABSTRACT

By using the fixed point theory on a cone with a special norm and space, we discuss the existence of positive solutions for a class of semipositone boundary value problems on infinite intervals. The work improves many known results including singular and non-singular cases.

1. Introduction

The main purpose of this paper is to study the existence of positive solutions for the following nonlinear singular and semipositone boundary value problem (BVP) on infinite intervals:

$$\begin{cases} (p(t)x'(t))' + \lambda f(t,x(t)) = 0, \ t \in (0,+\infty), \\ \alpha_1 x(0) - \beta_1 \lim\limits_{t\to 0^+} p(t)x'(t) = \sum\limits_{i=1}^{m-2} \gamma_i x(\eta_i), \\ \alpha_2 \lim\limits_{t\to+\infty} x(t) + \beta_2 \lim\limits_{t\to+\infty} p(t)x'(t) = \sum\limits_{i=1}^{m-2} \delta_i x(\eta_i), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\alpha_1, \ \alpha_2, \ \gamma_i, \ \delta_i \geqslant 0, \beta_1, \ \beta_2 > 0, \eta_i \in (0,+\infty), (1,2,\ldots,m-2)$ are given constants, $f : (0,+\infty)\times(0,+\infty) \to (-\infty,+\infty)$ is a continuous function and $f(t,u)$ may be singular at $t = 0$ and $u = 0$, $p \in C[0,+\infty) \cap C^1(0,+\infty)$ with $p > 0$ on $(0,+\infty)$, $\int_0^\infty \frac{1}{p(s)} ds < +\infty$, $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0,\infty) > 0$ in which $B(t,s) = \int_t^s \frac{1}{p(v)} dv$.

Boundary value problems on infinite intervals arises from the study of many real world problems such as solution of nonlinear elliptic equations and modeling of gas pressure in a semi-infinite porous medium. A great deal of work has been done in these areas such as those in [1–22] and the references therein. Over the last couple of decades, a great deal of results have been developed for differential and integral boundary value problems. O'Regan et al. studied in [23] the BVP

$$\begin{cases} x''(t)) + \varphi(t)f(t,x(t)) = 0, \quad t \in (a,+\infty), \\ x(a) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $f : [a,+\infty) \times (0,+\infty) \to [0,+\infty)$ is a continuous function and $\varphi : (a,+\infty) \to [0,+\infty)$ is continuous. The existence of multiple unbounded positive solutions is discussed using the theory of fixed point index. By using the fixed point theorem and the monotone iterative technique, Zhang [24] studied the problem

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$$\begin{cases} x''(t)) + q(t)f(t,x(t)) = 0, \; t \in (0,+\infty), \\ x(0) = \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \quad \lim_{t\to+\infty} x'(t) = x_\infty \geqslant 0, \end{cases}$$

where $\gamma_i \geqslant 0$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < +\infty$, $\sum_{i=1}^{m-2}\gamma_i < 1$, $f : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function and $q : (0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable function. Liu et al. in [25] established the existence of positive solutions for the following equation on infinite intervals by applying the fixed point theorem of cone map

$$\begin{cases} (p(t)x'(t))' + m(t)f(t,x(t)) = 0, \; t \in (0,+\infty), \\ \alpha_1 x(0) - \beta_1 \lim_{t\to 0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim_{t\to+\infty} x(t) + \beta_2 \lim_{t\to+\infty} p(t)x'(t) = 0, \end{cases}$$

in which $f : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $m : (0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable function and may be singular at $t = 0$. Also, by the use of the Krasnosel-skii fixed point theorem, upper and lower solutions, Schauder point theorem and inequality technique, Xing et al. in [26], Lian et al. in [27,28], Li and Zhao in [29] studied many equations with infinite intervals. However, all of the above studies are limited to the cases in which the nonlinear term is positive.

Inspired by the work of the above papers and many known results, in this paper, we study the existence of positive solutions to BVP (1.1), where $x \in C[0,+\infty)$ is said to be a positive solution of BVP (1.1) if and only if x satisfies (1.1) and $x(t) > 0$ for any $t \in [0,+\infty)$. By using the fixed point theory on a cone, some new existence results are obtained for the case where the nonlinearity is allowed to be sign changing and has singularity. We should address here that our work presented in this paper has various new features. Firstly, our study is on singular nonlinear differential boundary value problems, that is, $f(t,u)$ is allowed to be singular at $t = 0$ and $u = 0$, which leads to many difficulties in analysis. Secondly, the techniques used in this paper are the approximation method; and a special cone in a special space is established to overcome the difficulties caused by singularity and infinite interval. Also we find another substitute function to solve the problem associated with the semi-positive property of the nonlinear function. Thirdly, we discuss the boundary value problem with multi-point boundary conditions, that is, BVP (1.1) includes two-point and three-point boundary value problems as special cases. To our knowledge, the theory of Sturm–Liouville multi-point boundary value problems on infinite interval is yet to be developed. Our model improves and generalizes the results of previous papers to some degree.

The plan of the paper is as follows. In Section 2, we present the preliminaries and necessary lemmas that are to be used to prove our main results. The main results are given in Section 3, including results for a completely continuous operator, and the conditions for the existence of positive solutions for the BVP (1.1). In Section 4, an example is given to demonstrate the application of our theoretical results.

2. Preliminaries and lemmas

For convenience, we let

$$\begin{aligned} &a(t) = \beta_1 + \alpha_1 B(0,t), \quad b(t) = \beta_2 + \alpha_2 B(t,\infty), \\ &a(\infty) = \lim_{t\to+\infty} a(t) = \beta_1 + \alpha_1 B(0,\infty) < +\infty, \quad a(0) = \lim_{t\to 0} a(t) = \beta_1, \\ &b(\infty) = \lim_{t\to+\infty} b(t) = \beta_2, \quad b(0) = \lim_{t\to 0} b(t) = \beta_2 + \alpha_2 B(0,\infty) < +\infty, \end{aligned}$$

$$\Delta = \begin{vmatrix} \rho - \sum_{i=1}^{m-2}\gamma_i b(\eta_i) & \sum_{i=1}^{m-2}\gamma_i a(\eta_i) \\ \sum_{i=1}^{m-2}\delta_i b(\eta_i) & \rho - \sum_{i=1}^{m-2}\delta_i a(\eta_i) \end{vmatrix}.$$

It is obvious that $a(t)$ is increasing and $b(t)$ is decreasing on $[0,+\infty)$. Define

$$G(t,s) = \frac{1}{\rho} \begin{cases} a(s)b(t), & 0 \leqslant s \leqslant t < +\infty, \\ a(t)b(s), & 0 \leqslant t \leqslant s < +\infty. \end{cases} \tag{2.1}$$

Denote $\tau(t) = a(t)b(t)$, then for any $0 \leqslant t,s < +\infty$, we get

$$\begin{aligned} &0 \leqslant G(t,s) \leqslant G(s,s) \leqslant \frac{b(0)a(s)}{\rho}, \quad 0 \leqslant G(t,s) \leqslant \frac{\tau(t)}{\rho}, \\ &\overline{G}(s) = \lim_{t\to+\infty} G(t,s) = \frac{\beta_2 a(s)}{\rho} \leqslant G(s,s) < +\infty. \end{aligned} \tag{2.2}$$

Lemma 2.1. *Suppose* $\theta = \frac{1}{a(\infty)b(0)}$, *then* $G(t,s) \geqslant \theta\tau(t)G(s,s)$, $0 \leqslant t,s < +\infty$.