Applied Mathematics and Computation 227 (2014) 274–290

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Modeling of biological population problems using the element-free kp-Ritz method

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ARTICLE INFO

Keywords:
Element-free kp-Ritz method
Biological population equation
Parabolic equation

ABSTRACT

A degenerate parabolic equation arising in the spatial diffusion of biological population is analyzed using the mesh-free kp-Ritz method. The mesh-free kernel particle estimate is employed to approximate the 2D displacement field. A system of discrete equations is obtained through application of the Ritz minimization procedure to the energy expressions. To validate the accuracy of the results and stability of the present method, convergence studies were carried out based on influences of support size and number of nodes. Effectiveness of the mesh-free kp-Ritz method for biological population model is investigated by numerical examples in this paper. The present results were compared with results reported in extant literature and were found to be in good agreement with the literature.

1. Introduction

The biological population problems have attracted much attention and research recently. Biologists believe that dispersal or emigration play a key role in the regulation of population of some species. The diffusion of a biological species in a region B is described by the following three functions of position $\boldsymbol{x} = (x, y)$ in B and t: $\rho(x, t)$ denotes the population density, $v(x, t)$ denotes the diffusion velocity and $\sigma(x, t)$ denotes the population supply [1]. Consider the following degenerate parabolic equation arising in the spatial diffusion of biological populations

$$\rho_t = \rho_{xx}^2 + \rho_{yy}^2 + \sigma(\rho), \quad t > 0, \quad x, y \in R \tag{1}$$

with a given initial condition $\rho(x, y, 0)$, where ρ denotes the population density and σ represents the population supply due to births and net of deaths.

The field $\rho(x, t)$ gives the number of individuals, per unit volume, at position $\boldsymbol{x}$ and time t and its integral over any subregion R gives the total population of R at time t. The field $\sigma(x, t)$ gives the rate at which individuals are supplied (per unit volume) directly at $\boldsymbol{x}$ by births and deaths. The flow of population from point to point is described by the diffusion velocity $v(x, t)$, which represents the average velocity of individuals who at time t occupy $\boldsymbol{x}$. The fields ρ, v and σ must be consistent with the following law of population balance: for every regular subregion R of B and for all time t:

$$\frac{d}{dt}\int_R \rho dV + \int_{\partial R} \rho \cdot v \cdot \boldsymbol{n} dV = \int_R \sigma dV \tag{2}$$

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.11.033

where $\boldsymbol{n}$ is the outward unit normal to the boundary ∂R of R. This equation asserts that the rate of change of population of R plus the rate at which individuals leave R across its boundary must be equal to the rate at which individuals are supplied directly to R [1–3].

For general degenerate parabolic equations of Eq. (1), several papers have considered the existence, uniqueness and regularity of weak solutions [4–10]. Numerical solutions of the biological population equation have seldom been explored and investigated, though Mehdi Dehghan has done some numerical work in this field [11].

To avoid mesh generation, meshless techniques have attracted the attention of researchers in recent years. In a meshless (mesh free) method a set of scattered nodes are used instead of meshing the domain of the problem. For many engineering problems, such as large deformation and crack growth, it is necessary to deal with extremely large deformations or fractures of the mesh with the re-meshing technique. The meshless method is a new and interesting numerical technique that can solve many engineering problems that are not suited to conventional numerical methods, with minimum or no meshing at all [12]. Important meshless methods have been developed, such as smooth particle hydrodynamics methods (SPH) [13], radial basis function (RBF) [14–17], element free Galerkin method (EFG) [18], meshless local Petrov–Galerkin method (MLPG) [19–21], reproducing kernel particle method (RKPM) [22–31] and so on.

The Ritz [32] approximation approach, developed almost a century ago, is a generalization of the Rayleigh [33] method. It is based on the principle that a resonant vibrating system completely interchanges its kinetic and potential energy forms. In the Rayleigh method, a single trial function for the mode shape satisfying at least the geometric boundary conditions is employed, and then by equating the maximum kinetic and potential energies, an upper bound frequency solution is obtained. The Ritz or Rayleigh–Ritz method is a proven approximation technique used in computational mechanics; notable works include Kitipornchai et al. [34], Liew et al. [35,36], Cheung and Zhou [37,38], Zeng and Bert [39], Su and Xiang [40], Lim and Liew [41] and Liew and Feng [42].

The element-free kp-Ritz method is firstly developed and implemented for the free vibration analysis of rotating cylindrical panels by Liew et al. [43]. In the kp-Ritz method, the mesh-free kernel particle estimate is employed to approximate the displacement field and a system of discrete equations is obtained through application of the Ritz minimization procedure to the energy expressions. The shape functions in the kp-Ritz method are constructed entirely in terms of discrete nodes, mesh generation and mesh distorsion are avoided. In the conventional Ritz method, it is difficult to choose the appropriate trial functions for problems with complicated boundary conditions. The kp-Ritz approach overcomes this shortcoming by using only one shape function type to describe the interior domain. The boundary conditions can be imposed easily. After that, the kp-Ritz method was widely applied and used in all kinds of problems, such as free vibration of two-side simply-supported laminated cylindrical panels [44], nonlinear analysis of laminated composite plates [45], free vibration analysis of conical shells [46], postbuckling analysis of laminated composite plates [47], free vibration analysis of conical shell panels [48], dynamic stability analysis of composite laminated cylindrical shells [49], geometrically nonlinear analysis of cylindrical shells [50], dynamic stability analysis of composite laminated cylindrical panels [51], free vibration analysis of functionally graded plates [52], geometrically nonlinear analysis of functionally graded plates [53], thermoelastic analysis of functionally graded plates [54], Sine–Gordon equation [55], and 3D wave equation [56].

This paper employs the kp-Ritz method to study a degenerate parabolic equation arising in the spatial diffusion of biological populations. In this study, the displacement field is approximated as these kernel functions, an energy formulation is formulated and a system of discrete equations is obtained by the Ritz minimization procedure. In this kp-Ritz method, a standard weight function is employed to express the interior field, and the boundary conditions are enforced by the penalty method. A few selected problems are used as examples to study the applicability of the method and the numerical results are compared with the existing exact solutions.

2. kp-Ritz method for the biological population problems

2.1. Energy formulations

Considering the degenerate parabolic equation arising in the spatial diffusion of biological populations, i.e. Eq. (1) with the initial condition $\rho(x, y, 0) = \rho_0$ and the weighted integral form of Eq. (1) is obtained as follows

$$\int_\Omega w \cdot [\rho_t - \rho^2_{xx} - \rho^2_{yy} - \sigma(\rho)]d\Omega = 0 \tag{3}$$

The weak form of Eq. (3) is obtained as follows

$$\int_\Omega \delta w \cdot \rho_t d\Omega - \int_\Omega \delta w \cdot \nabla^2(\rho^2)d\Omega - \int_\Omega \delta w \cdot \sigma(\rho)d\Omega - \int_S w\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial x}n_y + \frac{\partial u}{\partial z}n_z\right)dS = 0 \tag{4}$$

where (n_x, n_y, n_z) is the unit normal vector.

By the subsection integration, we can get

$$\int_\Omega \delta w \cdot \rho_t d\Omega + \int_\Omega \delta(\nabla w)^T \cdot \nabla(\rho^2)d\Omega - \int_\Omega \delta w \cdot \sigma(\rho)d\Omega = 0 \tag{5}$$