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Step method for a system of integral equations from biomathematics

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ABSTRACT

Using the step method, we study the following system of integral equations from biomathematics

$$x(t) = g(t, x(t), x(t-\tau)) + \int_{t-\tau}^{t} f(s, x(s))ds, \quad t \in [a,b], \tau > 0$$

and we prove the existence, uniqueness and the convergence of the successive approximation sequence using the Perov contraction principle and step method with a weaker Lipschitz condition. Also, we propose a new algorithm of successive approximation sequence generated by the step method and we give an example to illustrate the applications of the abstract results.

1. Introduction

We consider the system of nonlinear integral equations from biomathematics

$$\mathbf{x}(t) = g(t, \mathbf{x}(t), \mathbf{x}(t-\tau)) + \int_{t-\tau}^{t} \mathbf{f}(s, \mathbf{x}(s))ds, \quad t \in [a,b], \tag{1.1}$$

$$\mathbf{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [a-\tau, a] \tag{1.2}$$

where $(\mathbb{B}, |\cdot|)$ is a Banach space, $\mathbf{g} \in C([a,b] \times \mathbb{B}^m \times \mathbb{B}^m, \mathbb{B}^m), \mathbf{g} = (g_1, \ldots, g_m), \mathbf{f} \in C([a,b] \times \mathbb{B}^m, \mathbb{B}^m), \mathbf{f} = (f_1, \ldots, f_m)$, $\boldsymbol{\varphi} \in C([a-\tau, a], \mathbb{B}^m), \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m)$ and $\tau > 0$ is a parameter.

In the one-dimensional case, the equations of this type were studied by Cooke and Kaplan [3], Ambro [1], Dobriţoiu [4], Dobriţoiu et al. [5], Guo and Lakshmikantham [8], Kirr [10], Precup and Kirr [13], Torrejon [26], Rus [16,15]. Also, the systems of integral equations were studied by Cañada and Zertiti [2], Dobriţoiu [6], Dobriţoiu and Şerban [7].

In the present work we use the ideas of Rus [14] to obtain existence, uniqueness theorems and the convergence of a iterative algorithm using Perov contraction principle, fibre contraction principle and step method. Such kind of results have been proved in [21] and [9] in the case of integro-differential equations with lags.

2. Fibre weakly Picard operator

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the terminologies and notations from [18]. For the convenience of the reader we shall recall some of them.

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Denote by $A^0 := 1_X, A^1 := A, A^{n+1} := A \circ A^n, n \in \mathbb{N}$, the iterate operators of the operator A. Also

$$P(X) := \{Y \subseteq X | Y \neq \emptyset\},$$
$$F_A := \{x \in X | A(x) = x\},$$
$$I(A) := \{Y \in P(X) | A(Y) \subseteq Y\}.$$

Definition 2.1. $A : X \to X$ is called a Picard operator (briefly PO) if:

(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 2.2. $A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n \in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

Definition 2.3. A matrix $Q \in \mathbb{R}_+^{m\times m}$ is called a matrix convergent to zero iff $Q^k \to 0$ as $k \to +\infty$.

Theorem 2.1 *(see [12,19,24]). Let $Q \in \mathbb{R}_+^{m\times m}$. The following statements are equivalent:*

(i) *Q is a matrix convergent to zero;*
(ii) *$Q^k x \to 0$ as $k \to +\infty, \forall x \in \mathbb{R}^m$;*
(iii) *$I_m - Q$ is non-singular and*

$$(I_m - Q)^{-1} = I_m + Q + Q^2 + \dots$$

(iv) *$I_m - Q$ is non-singular and $(I_m - Q)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C}, \det(Q - \lambda I_m) = 0$ imply $|\lambda| < 1$;*
(vi) *there exists at least one subordinate matrix norm such that $\|Q\| < 1$.*

The matrices convergent to zero were used by Perov [11] to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$.

Definition 2.4 (*see [11,19,20]*). Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}_+^m$ and $A : X \to X$. The operator A is called a Q-contraction if there exists a matrix $Q \in \mathbb{R}_+^{m\times m}$ such that:

(i) Q is a matrix convergent to zero;
(ii) $d(A(x), A(y)) \leqslant Qd(x, y)$, for all $x, y \in X$.

Theorem 2.2 *(Perov (see [19,20])). Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}_+^m$ and $A : X \to X$ be a Q-contraction. Then:*

(i) *A is PO, $F_A = F_{A^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$;*
(ii) *$d(A^n(x), x^*) \leqslant (I_m - Q)^{-1} Q^n d(x, A(x))$, for all $x \in X$.*

Theorem 2.3 *(Rus [17] (Generalized Fiber Contraction Theorem)). Let (X, d) be a metric space (generalized or not) and (Y, ρ) be a complete generalized metric space ($\rho(x, y) \in \mathbb{R}^m$). Let $A : X \times Y \longrightarrow X \times Y, A(x, y) = (B(x), C(x, y))$, be a continuous operator, where $B : X \longrightarrow X$ and $C : X \times Y \longrightarrow Y$ be two operators. We suppose that:*

(i) *B is a WPO;*
(ii) *There exists a matrix $Q \in \mathbb{R}_+^{m\times m}$, converging to zero, such that*

$$\rho(C(x, y_1), C(x, y_2)) \leqslant Q\rho(y_1, y_2),$$