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A new perturbation technique in solution of nonlinear differential equations by using variable transformation

N. Elmas *, H. Boyaci

Department of Mechanical Engineering, Celal Bayar University, 45140 Muradiye, Manisa, Turkey

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ABSTRACT

A perturbation algorithm using a new variable transformation is introduced. This transformation enables control of the independent variable of the problem. The problems are solved with new transformation: Classical Duffing equation with cubic nonlinear term and a singular perturbation problem. Results of multiple scales, Lindstedt Poincare method, new method and numerical solutions are contrasted.

1. Introduction

Perturbation method is one of the mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly. Perturbation theory can be applied if the problem can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem. Perturbation method leads to an expression for the desired solution in terms of a formal power series in some "small" parameter known as a perturbation series that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem.

Algebraic equations, integrals, differential equations, difference equations and integro-differential equations can be solved approximately with perturbation techniques. The direct expansion method (pedestrian expansion) does not produce physically valid solutions for most of the cases and depending on the nature of the equation, many different perturbation techniques such as Lindstedt–Poincare technique, renormalization method, method of multiple scales, averaging methods, method of matched asymptotic expansions etc. are developed within time [1].

Nonlinear ordinary differential equations by perturbation techniques are a quite common method [2–10]. However, the process may first discussed by Nayfeh et al. [2]. They consider the nonlinear dynamic response of a relief valve and showed the discrepancies between two methods for this specific problem.

In this study, a perturbation algorithm using a new variable transformation is introduced. The problems are solved with new transformation: Classical Duffing equation with cubic nonlinear term and a singular perturbation problem. Results of multiple scales, Lindstedt Poincare method, new method and numerical solutions are contrasted.

2. Solution of differential equations by perturbation technique using any time transformation

In direct perturbation Method, mostly secular terms appear of higher orders of the expansion invalidating the solution. In order to avoid this problem a new time transformation has been proposed in our study.

The new time transformation is defined as,

* Corresponding author.
E-mail address: nedret.elmas@cbu.edu.tr (N. Elmas).

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$$T_e = f(t, w, \varepsilon) \cdot t. \tag{1}$$

Using the chain rule, we transform the derivate accordingly

$$\frac{du}{dt} = \frac{du}{dT_e}\frac{dT_e}{dt} = \frac{du}{dT_e}\left(\underbrace{\frac{df}{dt}t + f}_{\frac{dT_e}{dt}}\right) = u'(\dot{f}t + f) \quad , \tag{2}$$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(\underbrace{\frac{du}{dT_e}\left(\frac{df}{dt}t + f\right)}_{\frac{du}{dt}}\right) = \left[\frac{d}{dT_e}\left(\frac{du}{dT_e}\right)\right]\left(\frac{df}{dt}t + f\right)\frac{dT_e}{dt} + \frac{du}{dT_e}\left[\frac{d}{dt}\left(\frac{df}{dt}t + f\right)\right]$$

$$= \frac{d^2u}{dT_e^2}\left(\frac{df}{dt}t + f\right)^2 + \frac{du}{dT_e}\left(\frac{d^2f}{dt^2}t + 2\frac{df}{dt}\right)$$

$$\frac{d^2u}{dt^2} = u''(\dot{f}t + f)^2 + u'(\ddot{f}t + 2\dot{f}). \tag{3}$$

So we have obtained a more effective time expression T_e without losing the original time parameter t using the function f. Thus, speeding up and slowing down control of the time parameter will be available as in method of multiple scales.

In Eq. (2) and (3) first order time-derivatives according to new time variable T_e appear in second order time derivative expressions according to original time variable (t). So, we are able to have information about some parameters of nonlinear differential equation, and to interpret the results.

By this new time transformation we have the advantages of both Lindstedt–poincare method and method multiple scales.

3. Duffing equation

Here, we will consider the cubic nonlinear Duffing equation with ε parameter [1–3].

$$\ddot{u} + u + \varepsilon u^3 = 0 \tag{4}$$

As initial conditions, we take.

$$u(0) = a_0 \quad \text{and} \quad \dot{u}(0) = 0 \tag{5}$$

The solution u of our problem is a function of the independent variable t and the parameter ε. We write $U = u(t;\varepsilon)$ where the parameter ε is separated from the independent variable t by a semicolon. In the next part, we determine a straightforward approximation to (4) and (5) for small but finite ε.

Approximate solutions are obtained with the Straightforward Expansion technique as.

$$u = a_0\cos(t + \beta_0) + \varepsilon\left[a_1\cos(t + \beta_1) - \frac{3}{8}a_0^3 t\sin(t + \beta_0) + \frac{1}{32}a_0^3\cos(3t + 3\beta_0)\right] \tag{6}$$

with Lindstedt–Poincare technique ($\tau = wt$, $w = 1 + \varepsilon\frac{3}{8}a^2$) as,

$$u = a\cos\left((1 + \varepsilon\frac{3}{8}a^2)t + \beta\right) + \varepsilon\left[\frac{1}{32}a^3\cos\left(3\left(1 + \varepsilon\frac{3}{8}a^2\right)t + 3\beta\right)\right] \tag{7}$$

and with method of multiple scales as,

$$u = a_0\cos(wt) + \frac{\varepsilon a_0^3}{32w_0^2}(\cos(3wt) - \cos(wt)) + O(\varepsilon^3) \tag{8}$$

where $w = 1 + \varepsilon\frac{3}{8}a^2$.

4. An application of the method developed in the study for nonlinear differential equations

At first, we will consider the well-known Duffing equation (4), for it is solution with new time transformation. Dependent variable and function of new time transformation are expanded as shown below.

$$u = u_0(T_e) + \varepsilon u_1(T_e) + \cdots \tag{9}$$

$$f = f_0(w,t) + \varepsilon f_1(w,t) + \cdots$$

$$\dot{f} = \dot{f}_0(w,t) + \varepsilon\dot{f}_1(w,t) + \cdots \ddot{f} = \ddot{f}_0(w,t) + \varepsilon\ddot{f}_1(w,t) + \cdots \tag{10}$$