Applied Mathematics and Computation 227 (2014) 456–468

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On exponential stability of nonlinear fractional multidelay integro-differential equations defined by pairwise permutable matrices

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ARTICLE INFO

Keywords:
Integro-differential equation
Multiple delays
Exponential stability

ABSTRACT

In this paper, systems of nonlinear differential equations with Caputo fractional derivative and multiple delays are considered. Using representation of a solution of differential equation with multiple delays in the form of matrix polynomial and stability results such as Gronwall's and Pinto's inequality, sufficient conditions for the exponential stability of a trivial solution of nonlinear multidelay fractional differential equations are proved.

1. Introduction

Recently, Medved' and Pospíšil extended in [17] the results from [10] for differential equations with a single delay to differential equations with multiple delays, assuming the linear parts to be given by pairwise permutable matrices. In the same paper, they used the derived matrix representation of the solution to establish the sufficient conditions for the exponential stability of the trivial solution of a system of differential equations with multiple delays, analogically to systems with a single delay (cf. [19]). Let us recall their result on the matrix representation of a solution.

Theorem 1. *Let $n \in \mathbb{N}$, $\tau_1, \ldots, \tau_n > 0$, $\tau := \max\{\tau_1, \ldots, \tau_n\}$, $\varphi \in C^1_\tau := C^1([-\tau, 0), \mathbb{R}^N)$, $A, B_1, \ldots, B_n$ be $N \times N$ pairwise permutable matrices, i.e. $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i, j \in \{1, \ldots, n\}$ and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. The solution of the Cauchy problem*

$$\dot{x}(t) = Ax(t) + B_1x(t - \tau_1) + \cdots + B_nx(t - \tau_n) + f(t), \tag{1.1}$$

$$x(t) = \varphi(t), \quad -\tau \leqslant t \leqslant 0 \tag{1.2}$$

has the form

$$x(t) = Y(t + \tau)\varphi(-\tau) + \int_{-\tau}^{0} Y(t - s)(\varphi'(s) - A\varphi(s))ds - \sum_{i=1}^{n} B_i \int_{-\tau}^{-\tau_i} Y(t - \tau_i - s)\varphi(s)ds + \int_0^t Y(t - s)f(s)ds \tag{1.3}$$

where

$$Y(t) = e^{At} e^{\widetilde{B}_1, \ldots, \widetilde{B}_n(t - \tau_n)}_{\tau_1, \ldots, \tau_n}, \qquad \widetilde{B}_i = e^{-A\tau_i}B_i, \qquad i = 1, \ldots, n.$$

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http://dx.doi.org/10.1016/j.amc.2013.11.012

Here $e_{\tau_1,\dots,\tau_n}^{B_1,\dots,B_n t}$ is the so-called multi-delayed matrix exponential corresponding to delays $\tau_1,\dots,\tau_n$ and matrices $B_1,\dots,B_n$ defined by

$$e_{\tau_1,\dots,\tau_n}^{B_1,\dots,B_n t} = \begin{cases} \Theta, & t < -\tau_n, \\ X_{n-1}(t+\tau_n), & -\tau_n \leqslant t < 0, \\ X_{n-1}(t+\tau_n) + B_n \int_0^t X_{n-1}(t-s_1)X_{n-1}(s_1)ds_1 + \cdots & \\ \quad \cdots + B_n^k \int_{(k-1)\tau_n}^t \int_{(k-1)\tau_n}^{s_1} \cdots \int_{(k-1)\tau_n}^{s_{k-1}} X_{n-1}(t-s_1) & \\ \times \prod_{i=1}^{k-1} X_{n-1}(s_i - s_{i+1}) X_{n-1}(s_k - (k-1)\tau_n) ds_k \dots ds_1, & (k-1)\tau_n \leqslant t < k\tau_n, k \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $X_{n-1}(t) = e_{\tau_1,\dots,\tau_{n-1}}^{B_1,\dots,B_{n-1}(t-\tau_{n-1})}$ and

$$e_\tau^{Bt} = \begin{cases} \Theta, & t < -\tau, \\ E, & -\tau \leqslant t < 0, \\ E + Bt + B^2 \frac{(t-\tau)^2}{2} + \cdots + B^k \frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leqslant t < k\tau, \ k \in \mathbb{N} \end{cases}$$

is the delayed matrix exponential from [10]. In the whole paper, Θ and E denote the zero and the identity matrix, respectively. Moreover, we always assume that $\|E\| = 1$.

We note that similar matrix representation of a solution was derived for difference equations with one delay [4], multiple delays [18], delayed oscillators [9] and functional differential equations [23]. Such form of solutions led to results in controllability theory [5,11], stability theory [17–20], boundary value problems [1–3], etc.

Of course, the idea to study stability of fractional differential equations is not a new one. Recently, different types have been discussed such as Mittag–Leffler stability [12], generalised Mittag–Leffler stability [13], Ulam stability [6,25], Ulam–Hyers stability [7,8], etc. We note that more references can be found in an overview paper [14]. Also finite-time stability and Lyapunov stability of fractional differential equations with retarded argument of the form

$${}^C D^\alpha x(t) = K_1 x(t-\tau_1) + \cdots + K_n x(t-\tau_n)$$

have been investigated using Laplace transform and final value theorem, or Lambert W function (for references see [14]). Here

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \dot{f}(s) ds$$

is Caputo derivative [22] of C^1-function f.

In the present paper, we focus on the exponential stability of the trivial solution of integro-differential equation

$${}^C D^\alpha (h(t)(\dot{x}(t) - Ax(t) - B_1 x(t-\tau_1) - \cdots - B_n x(t-\tau_n))) = F(t) \tag{1.5}$$

as an equivalent form (see Proposition 2) of a weakly nonhomogeneous linear equation (or weakly nonlinear in case of $F(t) = F(x(t))$)

$$\dot{x}(t) = Ax(t) + B_1 x(t-\tau_1) + \cdots + B_n x(t-\tau_n) + \frac{1}{h(t)} \left(h(0)(\dot{x}(0^+) - A\varphi(0) - B_1\varphi(-\tau_1) - \cdots - B_n \varphi(-\tau_n)) + I^\alpha F(t) \right)$$

with exponentially decreasing term $\frac{1}{h(t)}$. Here I^α denotes Riemann–Liouville fractional integral [22]

$$I^\alpha F(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s) ds. \tag{1.6}$$

We consider different types of function F and always assume $\alpha \in (0,1)$. So, we extend results from [24] where this problem was considered for a single delay. Instead of Lambert W function, we use multi-delayed matrix exponential and Theorem 1 to find the representation of the solution and, consequently, to derive sufficient conditions for the exponential stability of the trivial solution of (1.5). Note that due to the form of (1.5) we do not need to use Laplace transform either.

The paper is outlined as follows. In Section 2 we consider Eq. (1.5) with F representing the nonlinear part of this equation independent of any delay. In Section 3, we add the dependency on delays to function F and investigate the exponential stability of the trivial solution of nonlinear differential equations with multiple delays. Finally, we provide a simple example to illustrate our results.

For the convenience of a reader we omit only some details in the proofs of our main results, where the steps repeat. By this, one can see how integral inequalities are applied and a desired stability result is obtained. Moreover, in Theorems 8 and 11 we state in the second condition particular cases of the first conditions (e.g. clearly if $f(x) = o(\|x\|^2)$ then also $f(x) = o(\|x\|)$ in the neighbourhood of 0). The reason is that, a better estimation of the nonlinear function can lead to a larger neighbourhood of 0 for the exponential stability. To estimate the size of the neighbourhood one would have to follow the particular proof and it would be important whether Pinto's or Gronwall's inequality is used. However, these computations exceed the scope of this paper. So we leave them for future research.