Applied Mathematics and Computation 227 (2014) 593–600

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Analytic Riemann boundary value problem on h-summable closed curves

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ARTICLE INFO

Keywords:
Analytic functions
Riemann boundary value problem
Fractal dimensions

ABSTRACT

The aim of this work is to further extend the notion of d-summability due to Harrison and Norton in the beginning of the 1990s. Explicit examples are given to illustrate how our notion can be applied to describe the geometry of a simply connected bounded open subset of $\mathbb{C}$ in a more delicate manner than the latter one. Applications on the solvability conditions for the Riemann boundary value problems for analytic functions over closed curves merely required to be summable in the generalized sense are described.

1. Introduction

The classical Riemann boundary value problem with Hölder continuous coefficients has been discussed in a number of special classes of domains. Many well-known results on its solvability, mainly concern the case of a piece-smooth boundaries, was described by Gakhov in [7] and by Lu in [18].

Generalizations of the Riemann boundary value problem has been investigated together with new theoretical results not only for non-smoothly bounded domain, which differs with the former, but for general assumptions on the data of the problem, such as generalized Hölder coefficients or special subspaces of it.

Simultaneously, this boundary value problem has attracted a growing interest in numerous applications in elasticity theory, hydro and aerodynamics, theory of orthogonal polynomials and so on.

Also, the Riemann boundary value problem was studied for generalized analytic functions, as well as for many other linear and nonlinear elliptic systems in the plane. The best general references here are [3,8].

During the last decades, new analytic toolkit to treat the analytic Riemann boundary value problems on strong non-rectifiable curves have arisen, cf. for example [1,2,11,12,14–17].

Our purpose is to describe a sufficiently complete picture of solvability of the analytic Riemann boundary value problem for a great generality dealing directly with the notion of h-summability (to be introduced later) of the boundary as essential hypothesis for integration. In the process of this study we find that basic results obtained in the above cited references are extended or improved to a more suitable context.

Throughout the paper we assume Ω to be a simply connected bounded open subset of $\mathbb{C}$ and γ is the boundary curve of Ω. When necessary we shall use the temporary notation $\Omega_+ := \Omega,\ \Omega_- := \mathbb{C} \setminus \overline{\Omega}$.

The analytic Riemann boundary value problem considered here is to find all functions Φ analytic in $\overline{\mathbb{C}} \setminus \gamma$ satisfying the boundary condition

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http://dx.doi.org/10.1016/j.amc.2013.11.053

$$\Phi^+(\tau) = F(\tau)\Phi^-(\tau) + f(\tau), \quad \tau \in \gamma, \tag{1}$$

where F and f are two given continuous functions defined on γ, and $\Phi^\pm(\tau)$ are the limit values of the desired function Φ at $\tau \in \gamma$ as this point is approached from $\Omega_\pm$ respectively. A simplest particular case of (1) is the so-called jump problem:

$$\Phi^+(\tau) - \Phi^-(\tau) = f(\tau), \quad \tau \in \gamma. \tag{2}$$

When investigating the above-stated problem (1), it is required that the unknown functions are continuous up to the boundary, for what is usually called the continuous analytic Riemann boundary value problem.

2. Preliminaries

In this section we describe the basic definitions and technical results used in the proofs of the main theorems of the paper.

2.1. Functional spaces in $\mathbb{C}$

We will denote by $\mathcal{C}(\mathbf{E})$ the set of all continuous complex functions defined on $\mathbf{E} \subset \mathbb{R}^2 \simeq \mathbb{C}$. If $\mathbf{E}$ is a bounded set of $\mathbb{C}$, then $\mathcal{H}_\varphi(\mathbf{E}) \subset \mathcal{C}(\mathbf{E})$ stands for the class of all generalized Hölder continuous functions f for which

$$\omega_f(\delta, \mathbf{E}) = \sup\{|f(x) - f(y)| : \ |x - y| \leqslant \delta, \ x, y \in \mathbf{E}\} \leqslant c\,\varphi(\delta),$$

where φ is a majorant, i.e., a defined, finite, positive, non-decreasing function in $(0, +\infty)$ with $\lim_{\delta \to 0+} \varphi(\delta) = 0$. For example, $\varphi(\delta) = \delta^v$, $\delta \in (0, |\mathbf{E}|]$, $0 < v \leqslant 1$, is a majorant and we have the usual Hölder class $\mathcal{H}_v(\mathbf{E}) \subset \mathcal{C}(\mathbf{E})$. Here and subsequently, $|\mathbf{E}|$ denotes the diameter of $\mathbf{E} \subset \mathbb{C}$.

It is worth pointing out that a very successful tool in the theory of Riemann boundary value problems for analytic functions is the Cauchy type integral. Hence, it is not surprising the necessity of discuss this concept in a full generality manner concerning the geometric properties of the integration contour.

In particular, if γ is a Jordan closed rectifiable curve, then for any $f \in \mathcal{C}(\gamma)$, the customary Cauchy type integral

$$(\mathbf{C}_\gamma f)(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \notin \gamma, \tag{3}$$

exists and represents an analytic function in $\overline{\mathbb{C}} \setminus \gamma$.

At almost all points $\tau \in \gamma$ this function has non-tangential boundary limit values from both sides, and these values almost everywhere satisfy the relation

$$(\mathbf{C}^+_\gamma f)(\tau) - (\mathbf{C}^-_\gamma f)(\tau) = f(\tau), \quad \tau \in \gamma. \tag{4}$$

If $f \in \mathcal{H}_v(\gamma)$ and $v > \frac{1}{2}$, then the function $\mathbf{C}_\gamma f$ has continuous boundary values on the whole γ as proved in [4,19].

When γ is assumed to be non-rectifiable then the definition (3) of the Cauchy type integral falls, but the analytic Riemann boundary value problem is still suitable and the influence of the geometry of the boundary on the solvability of the problem is necessarily reveled.

We know from [13] that this problem is solvable if the boundary data satisfy the Hölder condition with exponent $v > \frac{1}{2}\dim\gamma$, where $\dim\gamma$ is the box dimension (also known as Minkowski dimension) of the curve γ. In addition, the same author has introduced two new versions of the metric dimension of γ, the so-called approximate dimension and refined metric dimension (see [14,15]). It is proved that the replacement of the box dimension by one of these new characteristics improves this solvability condition.

In Section 3 we will look more closely at this phenomenon for a much more pathological situation.

We continue this section by introducing some important facts of fractal geometry.

2.2. Fractal dimensions and summable sets in $\mathbb{C}$

Let $\mathbf{E}$ be an arbitrary subset of $\mathbb{R}^2 \simeq \mathbb{C}$. Then for any $s \geqslant 0$ its Hausdorff measure $\mathcal{H}^s(\mathbf{E})$ may be defined by

$$\mathcal{H}^s(\mathbf{E}) = \lim_{\delta \to 0} \inf\left\{\sum_{k=1}^{\infty} (\operatorname{diam} B_k)^s : \ \mathbf{E} \subset \bigcup_{k=1}^{\infty} B_k, \ \operatorname{diam} B_k < \delta\right\}$$

where the infimum is taken over all countable δ-coverings $\{B_k\}$ of $\mathbf{E}$ with open or closed balls.

Now, let $\mathbf{E}$ be a compact subset of $\mathbb{R}^2$. The Hausdorff dimension of $\mathbf{E}$, denoted $\dim_H(\mathbf{E})$, is then defined as the infimum of all $s \geqslant 0$ such that $\mathcal{H}^s(\mathbf{E}) < \infty$. For more details concerning the Hausdorff measure and dimension we refer the reader to [6].

Frequently however, the so-called box dimension is more appropriated than the Hausdorff dimension to measure the roughness of a given set $\mathbf{E}$. By definition, the box dimension of a compact set $\mathbf{E} \subset \mathbb{R}^2$ is equal to

$$\dim \mathbf{E} := \limsup_{t \to 0} \frac{\log N_{\mathbf{E}}(t)}{-\log t}, \tag{5}$$