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On a delay population model with a quadratic nonlinearity without positive steady state

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ABSTRACT

A population model described by a nonlinear delay differential equation with a quadratic nonlinearity

$$\dot{x}(t) = \sum_{k=1}^{m} \alpha_k(t)x(h_k(t)) - \beta(t)x^2(t), \quad t \geqslant 0$$

is considered where $m \geqslant 1$ is an integer, functions $\alpha_k, \beta : [0,\infty) \to (0,\infty)$ are continuous, functions $h_k : [0,\infty) \to \mathbb{R}$ are continuous such that $t - \tau \leqslant h_k(t) \leqslant t, \tau = \text{const}, \tau > 0$, and, for any $t \geqslant 0$, the inequality $h_j(t) < t$ holds for at least one index $j \in \{1,\ldots,m\}$.

Although this equation does not have a positive steady state, a new method not based on the existence of a positive steady state is developed and used to investigate the permanence, global attractivity conditions and nonoscillation properties.

1. Introduction

In the monograph [1], the author considers the following population model with a quadratic nonlinearity

$$\dot{x}(t) = \sum_{k=1}^{m} \alpha_k x(t - \tau_k) - \beta x^2(t), \quad t \geqslant 0, \tag{1}$$

where α_k, β and τ_k are positive constants, and with the initial condition

$$x(t) = \varphi(t), \quad t \in [-\tau^*, 0],$$

where $\varphi : [-\tau^*, 0] \to (0,\infty)$ is a continuous function and $\tau^* = \max_{k=1,\ldots,m} \tau_k$. He proved that the positive equilibrium

$$K^* := \frac{1}{\beta} \cdot \sum_{k=1}^{m} \alpha_k \tag{2}$$

is a global attractor for all positive solutions of the Eq. (1). A similar result was obtained in [2] for $m = 1$.

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This result is different from almost all known results about stability for nonlinear delay differential equations since there are no limitations on the parameters entering Eq. (1).

In [3], we considered the equation

$$\dot{x}(t) = r(t)\left[\sum_{k=1}^{m}\alpha_k x(h_k(t)) - \beta x^2(t)\right], \quad t \geqslant 0, \tag{3}$$

where α_k and β are positive constants, $h_k : [0,\infty) \to \mathbb{R}$ are continuous functions such that inequalities $t - \tau \leqslant h_k(t) \leqslant t$ are valid for a $\tau = \text{const}, \tau > 0$, for any $t \geqslant 0$, the inequality $h_j(t) < t$ holds for at least one $j \in \{1,\ldots,m\}$, and $r : [0,\infty) \to (0,\infty)$ is a continuous function satisfying inequality $r(t) \geqslant r_0 = \text{const}$ for an $r_0 > 0$. We also proved that the positive equilibrium K^*, defined by (2), is a global attractor for all positive solutions of (3).

In [4] the authors consider the equation

$$\dot{x}(t) = \frac{\alpha x(t-\tau)}{1 + \beta x(t-\tau)} - \mu x(t) - \kappa x^2(t), \tag{4}$$

with constants $\alpha, \beta, \mu, \kappa$ and τ positive, which has asymptotic properties similar to those of Eqs. (1) and (3). In particular, the unique positive equilibrium of Eq. (4) attracts all positive solutions of this equation.

The following result was also obtained in [4]. If the initial function is greater (smaller) than the positive equilibrium, then the solution of Eq. (4) is also greater (smaller) than this equilibrium.

In the present paper, we will consider the equation

$$\dot{x}(t) = \sum_{k=1}^{m}\alpha_k(t)x(h_k(t)) - \beta(t)x^2(t), \quad t \geqslant 0, \tag{5}$$

where $\alpha_k, \beta : [0,\infty) \to (0,\infty)$ are continuous functions, for some positive constants α_0, A_0, β_0 and B_0 inequalities

$$\alpha_0 \leqslant \alpha_k(t) \leqslant A_0, \quad \beta_0 \leqslant \beta(t) \leqslant B_0$$

hold on $[0,\infty)$, functions $h_k : [0,\infty) \to \mathbb{R}$ are continuous such that, for a constant $\tau > 0$, we have

$$t - \tau \leqslant h_k(t) \leqslant t$$

and, for each $t \in [0,\infty)$,

$$\min_{j\in\{1,\ldots,m\}} h_j(t) < t. \tag{6}$$

Together with (5) we consider an initial problem

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{7}$$

where $\varphi : [-\tau, 0] \to (0,\infty)$ is a continuous function.

We will discuss the asymptotic properties of the problem (5) and (7). Unlike Eqs. (1), (3) and (4), (5) does not have a positive steady state. Such equations are more difficult to investigate than equations with a positive steady state and we note that there are not so many papers on equations with such a property. We can mention here a very interesting paper [5], in which the author considers a general delay logistic equation without positive steady state and gives a survey on some other papers with a similar property.

In this paper, we prove that the solution of problem (5) and (7) is positive and global. Let us recall that a function $x : [-\tau,\infty) \to \mathbb{R}$ continuous on $[-\tau,\infty)$ and continuously differentiable on $[0,\infty)$ is called a global solution of problem (5) and (7) if it satisfies Eq. (5) on $[0,\infty)$ and initial condition (7).

We also prove that Eq. (5) is uniformly permanent obtaining explicit eventually lower and upper bounds of all solutions. Finally, we obtain some nonoscillation properties of Eq. (5), similar to those obtained in [4] for Eq. (4).

We will use the following standard notions, defined below.

Definition 1. A number $K > 0$ is a global attractor for all solutions of Eq. (5) defined by all initial functions (7) if every solution of (5) and (7) is a global solution and $\lim_{t\to\infty} x(t) = K$.

Definition 2. Eq. (5) is called uniformly permanent if there exist two positive numbers $m, M, m < M$ such that, for all solutions $x(t)$ of Eq. (5) defined by all initial functions (7), we have

$$m \leqslant \liminf_{t\to\infty} x(t) \leqslant \limsup_{t\to\infty} x(t) \leqslant M.$$

2. Main results

Theorem 1. *Every solution of problem (5) and (7) is a positive and global solution.*