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An efficient numerical method for solving Abel integral equation

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ABSTRACT

In this paper, a numerical method for solving Abel integral equation is presented. We first convert the integral equation to the algebraic equation. Then we find numerical inversion of Laplace transform by power series. At last the Padé approximants is effectively used to improve the convergence rate and accuracy of the computed series. The method is described and illustrated with numerical examples. The results reveal that the method is accurate and easy to implement.

1. Introduction

Abel integral equations have many applications in various areas. For instance, mathematical physics, chemistry, electrochemistry, semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reaction and population dynamics [1,2].

The generalized Abel integral equations often appear in two forms the first and second kind as follows respectively

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha} dt \tag{1}$$

and

$$u(x) = f(x) + \int_0^x \frac{u(t)}{(x-t)^\alpha} dt, \tag{2}$$

where $0 < \alpha < 1, f(x) \in C[0,1], 0 \leqslant x, t \leqslant 1$. Our main aim is offering a new approach for solving the generalized cases.

Several numerical methods for approximating the solution of singular integral equation are known. Particularly, Huang [3] used the Taylor expansion of unknown function and obtained an approximate solution. Piessens [4] proposed a method to the solution of Abel integral equation using the Chebyshev polynomials. Yousefi [5] presented a numerical method for solution of Abel integral equation by Legendre wavelets. Moreover, Adomian decomposition method [6–8], the Homotopy perturbation method [8,9] and the Laplace decomposition method [10] have been proposed for obtaining the approximate analytic solution of Abel integral equation.

In this paper, first we reduce the singular Volterra integral equations to the algebraic equations by using the Laplace transform. Then applying the Taylor expansion and inverse Laplace transforms to the mentioned algebraic equations, a series that is uniformly convergent to the exact solution is obtained. The main advantage of the method is its simplicity, that is only a few of terms of the expansion are needed to get a good convergent numerical results.

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2. Laplace transform and their properties

In this section, we give some definitions and properties of Laplace transform.

The Laplace transform of a function $f(x), x > 0$ is defined as

$$\mathcal{L}[f(x)] = F(s) = \int_0^{+\infty} e^{-sx} f(x) dx,$$

where s can be either real or complex. It has following properties:

Linearity property:

$$\mathcal{L}[af(x) + bg(x)] = a\mathcal{L}[f(x)] + b\mathcal{L}[g(x)],$$

where a, b are constants.

Theorem 2.1 (*The convolution theorem*). *Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by*

$$\mathcal{L}[f_1(x)] = F_1(s), \quad \mathcal{L}[f_2(x)] = F_2(s).$$

The Laplace convolution product of these two functions is defined by

$$\mathcal{L}\left[\int_0^x f_1(x-t) f_2(t) dt\right] = F_1(s) F_2(s), \tag{3}$$

Theorem 2.2 [11]. *Suppose $F(s)$ is the Laplace transform of $f(t)$, which has a Maclaurin power series expansion in the form*

$$f(t) = \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}. \tag{4}$$

Taking the Laplace transform, it is possible to write formally

$$F(s) = \sum_{i=0}^{\infty} \frac{a_i}{s^{i+1}}. \tag{5}$$

Conversely, we derive (4) form a given expansion (5).

3. Padé approximant

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function. The $[L/M]$ Padé approximant to a formal power series $y(t) = \sum_{i=0}^{\infty} a_i t^i$ is given by:

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)} = \frac{p_0 + p_1 t + \cdots + p_L t^L}{1 + q_1 t + \cdots + q_M t^M}. \tag{6}$$

The two polynomials in the numerator and denominator of (6) have no common factor. This means that the formal power series

$$y(t) = \frac{P_L(t)}{Q_M(t)} + O(t^{L+M+1}).$$

In this case Padé approximant $[L/M]$ is unique determined.

4. Solution of Abel integral equation

In this section we solve Abel integral Eq. (1) and (2) by using Laplace transform and Taylor series. First, we apply the Laplace transform to both sides of (1)

$$\mathcal{L}[f(x)] = \mathcal{L}\left[\int_0^x \frac{u(t)}{(x-t)^{\alpha}}\right].$$

Using the property of Laplace transform (3), we get

$$\mathcal{L}[f] = \mathcal{L}[u]\mathcal{L}[x^{-\alpha}].$$

Thus the given equation is equivalent to