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Maximum norm a posteriori error estimates for a singularly perturbed differential difference equation with small delay ☆

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ABSTRACT

A singularly perturbed differential difference equation with small delay is discretized on an adaptive grid which is formed by equidistributing arc-length monitor function. We first derive first-order maximum norm a posteriori estimates for the full discretization scheme of these problems. Then a first-order rate of convergence, independent of the perturbation parameter and the small delay parameter, is established. Numerical results are provided that support our theoretical estimates.

1. Introduction

In this paper, we consider the following singularly perturbed differential-difference equation in the domain $\Omega = (0,1)$:

$$-\varepsilon u_\varepsilon''(x) - a(x)u_\varepsilon'(x-\delta) + b(x)u_\varepsilon(x) = f(x), \tag{1.1}$$

$$u_\varepsilon(x) = \phi(x),\ -\delta \leqslant x \leqslant 0, u_\varepsilon(1) = \gamma, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small parameter and the delay parameter δ is such that $0 < \delta \ll 1$, which is of $O(\varepsilon)$. The functions $a(x),\ b(x),\ f(x)$ and $\phi(x) \in C[0,1]$, and γ is a constant. These equations are known as DDE. Differential–difference equations (DDE) are widespread in many branches of sciences and have been used for many years in the biosciences, engineering and control theory, etc. (see [1]). If we restrict these equations to a class in which the highest derivative term is multiplied by a small parameter, then we get singularly perturbed delay differential equations of the retarded type.

Owing to the presence of steep layer in the solution of (1.1) and (1.2), it is difficult to approximate the solution efficiently by various numerical methods using uniform grid [2,3]. To obtain a reliable numerical solution, many authors think of solving the above problem by using layer-adapted mesh approach. Kadalbajoo and Ramesh [4] first analysed a simple upwind scheme, midpoint upwind scheme and a hybrid scheme, respectively, on a Shishkin mesh to approximate the solution of the problem (1.1) and (1.2), where the hybrid algorithm used central difference in the boundary layer region and midpoint upwind scheme outside the boundary layer. Then, they used the same method to solve the second-order differential equations in which the highest order derivative was multiplied by a small parameter ε and both the differentiated (convection) and undifferentiated (reaction) terms were with negative shift δ (see [5]). In [6], Patidar and Sharma used non-standard finite difference methods (NSFDMs) to solve the problem (1.1) and (1.2)), and they shew that these NSFDMs were ε-uniformly convergent. Kadalbajoo and Kumar [7,8] considered the linear and nonlinear case of problem (1.1) and (1.2). Using the B-spline

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collocation method, they obtained almost second-order parameter uniform convergence on a piecewise-uniform mesh. Rao and Chakravarthy [9] proposed an exponentially fitted tridiagonal finite difference method for problem (1.1) and (1.2). The method was shown to be second-order ε uniform convergent.

Another approach is the use of adaptive mesh generated by equidistributing a monitor function over the domain of the problem. Very recently, Mohapatra and Natesan [10,11] presented an upwind difference scheme for a class of singularly perturbed differential–difference equations on a grid which was formed by equidistributing arc-length monitor function. Their method was shown to be first-order accurate in the maximum norm. They studied the optimal order which was independent of the perturbation parameter for the problem (1.1) and (1.2) based on a semi-discretization approach. In this work, we will derive the maximum norm a posteriori error estimates for the full discretization scheme of (1.1) and (1.2). Then, by using the techniques developed in [12], we obtain the first-order rate of convergence for the presented adaptive grid scheme, independent of the perturbation parameter and the delay parameter.

Notations Throughout this paper we use C, sometimes subscripted, to denote a generic positive constant that is independent of the parameters ε, δ and mesh parameter N. It may take different values in different place. In our estimates, we use the L_∞, L_1 and the negative norms defined by

$$\|v(x)\|_\infty = \operatorname*{ess\,sup}_{x\in[0,1]} |v(x)|, \quad \|v(x)\|_1 = \int_0^1 |v(x)|dx, \quad \|v(x)\|_* = \min_{V:V'=v} \|V(x)\|_\infty.$$

A mesh function $\varphi := \{\varphi(t_i)\}_{i=0}^N$ is a real-valued function. Define the discrete maximum norm for such functions by $\|\varphi\|_\infty = \max_{i=0,1,\dots,N}|\varphi(t_i)|$.

2. Continuous problem

Assuming that $\delta = k\varepsilon > 0$, where k is sufficiently small, and using the technique as done in [13,14], we shall expand the delay arguments through Taylor's series expansion on the solution of (1.1) and (1.2), so that the problem (1.1) and (1.2) reduces to a standard singularly perturbed two-point boundary value problem. However, for large k, there may be oscillations in the solution that grow exponentially. Some methods are developed in [15,16] to solve such kind of problems. Thus, we do not consider the case for large k in this paper. Now, expanding the delay terms, we have

$$u'_\varepsilon(x-\delta) = u'_\varepsilon(x) - \delta u''_\varepsilon(x) + \cdots, \quad \text{as} \quad \varepsilon \to 0. \tag{2.1}$$

Using the first two terms of (2.1) in the differential equation given by (1.1) and (1.2), we can obtain the following singularly perturbed boundary value problem

$$\begin{cases} \mathcal{L}u(x) := -[\varepsilon - \delta a(x)]u''(x) - a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ u(x) = \phi(x), \ -\delta \leqslant x \leqslant 0, \\ u(1) = \gamma. \end{cases} \tag{2.2}$$

It is noted that the problem (2.2) is an approximate differential equation which is different from the original Eq. (1.1) and (1.2) by a term which is of $O(\varepsilon^2)$. Thus, in (2.2), we use $u(x)$ as a different notation for $u_\varepsilon(x)$. In addition, we have taken $\phi(x)$ as a constant (see [13,14]), and assume that

$$\overline{\alpha} \geqslant a(x) \geqslant \alpha, \ \overline{\beta} \geqslant b(x) \geqslant \beta, \ |b'(x)| \leqslant \overline{\beta}, \ |f(x)| \leqslant \overline{\gamma}$$

and $\varepsilon - a(x)\delta > 0, \forall x \in [0,1]$. Under these assumptions, there has a unique solution for the above problem (2.2) which has a boundary layer of thickness $O(\varepsilon)$ near the boundary $x = 0$. Let $G(x,\xi)$ be the Green's function associated with $\mathcal{L}$ and Dirichlet boundary conditions. Then the solution u of (2.2) is given by

$$u(x) = \int_0^1 G(x,\xi)f(\xi)d\xi. \tag{2.3}$$

The operator $\mathcal{L}$ satisfies a maximum principle (see [10] Lemma 2.1), which implies the following stability estimate:

Lemma 2.1 (*Stability result for continuous problem*). *Let $u(x)$ be the solution of problem (2.2), then*

$$\|u(x)\|_\infty \leqslant \max\{|u(0)|, |u(1)|\} + \frac{1}{\beta}\|\mathcal{L}u(x)\|_\infty \tag{2.4}$$

$$\|u(x)\|_\infty \leqslant \frac{2}{\alpha}\|\mathcal{L}u(x)\|_*. \tag{2.5}$$

Proof. The proof of (2.4) can be seen in Lemma 2.2 of [10].

For the Green's function we have the following bounds from [17]