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Ice formation in the Arctic during summer: False-bottoms

Phan Thanh Nam [a,b], Pham Ngoc Dinh Alain [a,*], Dang Duc Trong [b], Pham Hoang Quan [b]

[a] *Mathematics Department, MAPMO UMR 6628, BP 67-59, 45067 Orleans cedex, France*
[b] *Department of Mathematics, HoChiMinh City National University, Viet Nam*

A R T I C L E I N F O

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A B S T R A C T

The only source of ice formation in the Arctic during summer is a layer of ice called false-bottoms between an under-ice melt pond and the underlying ocean. Of interest is to give a mathematical model in order to determine the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. In one dimension, this problem may be considered mathematically as a two-phase Stefan problem with two free boundaries. Our main result is to prove the existence and uniqueness of the solution from the initial condition.

1. Introduction

There are some different reservoirs of fresh water in the Arctic during summer (see, e.g., Eicken et al. [2]). First, melt water collects in surface melt pond (melting under the sun) which is the most important reservoir. Second, this melt water can percolate into the ice matrix to form an *under-ice melt pond* (see [5] for more detail). At the interface between this fresh water and the underlying salt water, double-diffusive convection of heat and salt leads to the formation of a layer of ice called *false-bottoms* (see Fig. 1 below). Very early, Nansen [8] in 1897 noted that this is the only source of forming new ice in the Arctic during the summer. This phenomenon has been considered for a long time by many authors (see, e.g., [1,2,5–7,9–11]). However, it has been considered in geophysical view-point based on practical experiments rather than rigorously mathematical formulations.

One of the most interesting ones is the simultaneous growth and ablation of false-bottoms, which is governed by both of heat fluxes and salt fluxes. The ablation of the sea-ice interface is caused by dissolution rather than by melting. Note that salt water has the double properties: it does not freeze even for temperature less than 0 °C, and it dissolves ice when it is in contact with ice. McPhee et al. [7] emphasized that properly describing heat and salt flux at the ice-ocean interface is essential for understanding and modeling the false-bottoms, and in particular without the double diffusion at this interface false bottoms would be so short-lived. The growth of the upper interface between a false bottom and a under-ice melt pond is governed by the purely thermodynamic condition at the interface.

Recently, in 2003, Notz et al. [9] gave a model simulating successfully the simultaneous growth and ablation of false-bottoms. They formulated mathematically the problem by a system of partial differential equations and solved them numerically by using a numerical routine in Mathematica. Although their numerical result fits quite well to early experimental data from Martin and Kauffman [6], a rigorous proof of the existence and uniqueness of the solution for the system of equations is still unavailable. Our aim in this paper is to give a such a mathematical proof. More precisely, we shall represent the problem explicitly by a system of partial differential equations associated with free boundary conditions similar to [9], and then show that the system has a unique solution from given initial conditions.

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[*] Corresponding author.
E-mail addresses: alain.pham@univ-orleans.fr, alain.pham@math.cnrs.fr (P.N. Dinh Alain).

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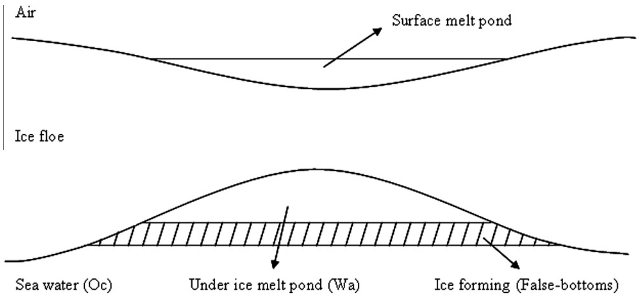

Fig. 1. Ice formation in the Arctic during summer.

Now we consider a one-dimensional model describing the simultaneous growth and ablation of the ice of false-bottoms. Here we have three environments: the ocean (Oc), the ice of false-bottoms (Fb) and the fresh water (Wa). Denote by $T(x,t), S(x,t)$ the temperature and the salinity, and denote by $h_0(t), h_u(t)$ the free boundaries at the interfaces ice-ocean (Fb–Oc) and ice-water (Fb–Wa), respectively (see Fig. 2 below).

At the interface Fb–Oc, we apply the first principle of thermodynamics, (i.e. variation of energy = variation of heat flux through the interface)

$$\Delta U = \Delta \Phi.$$

The net amount of heat transferred through the interface Fb–Oc in a section s is equal to

$$\Delta U = -dh_0 \rho_I L_f s,$$

where ρ_I is the density of the ice and L_f is the latent heat of fusion. On the other hand, the difference of the heat fluxes through a section s in the ice and the ocean during a time dt is

$$\Delta \Phi = (-\lambda_I T_x(h_0(t)+,t) + \lambda_O T_x(h_0(t)-,t))sdt,$$

where λ_I, λ_O are thermal conductivities of the ice and the ocean. Here the notations $h_0(t)+$ and $h_0(t)-$stand for the right limit and the left limit at $x = h_0(t)$. Thus the law of conservation of energy mentioned above, i.e. $\Delta U = \Delta \Phi$, leads to the Stefan condition for the heat balance at the interface

$$h_0'(t) = \widetilde{\lambda}_I T_x(h_0(t)+,t) - \widetilde{\lambda}_O T_x(h_0(t)-,t), \tag{1.1}$$

where

$$\widetilde{\lambda}_I = \frac{\lambda_I}{\rho_I L_f} > 0, \quad \widetilde{\lambda}_O = \frac{\lambda_O}{\rho_I L_f} > 0.$$

For simplicity, we can neglect the salt of the ice of false-bottoms. The water near the interface Fb–Oc is a mixture of melt water, which melts from the ice of false-bottoms, and sea water. This water freshens at the rate $S_0(t)h_0'(t)$, while salt diffuses into this water at the rate $-DS_x(h_0(t)-,t)$, where $S_0(t) = S(h_0(t)-,t)$ is the salinity of the ocean at the interface and $D > 0$ is the molecular diffusivity of salt in sea water. The balance of salt at this interface leads to the conservation condition

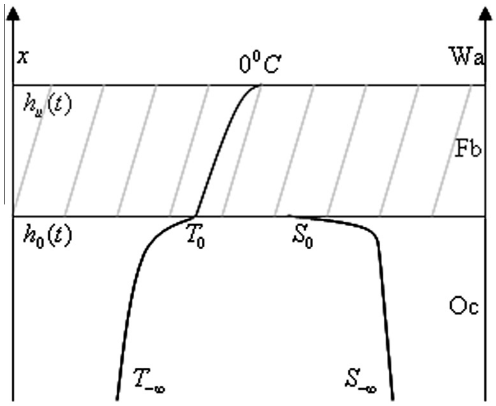

Fig. 2. One-dimensional model.