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Anticipated backward doubly stochastic differential equations

Xiaoming Xu

Institute of Finance and Statistics, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

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ABSTRACT

In this paper, we deal with a new type of differential equations called anticipated backward doubly stochastic differential equations (anticipated BDSDEs). The coefficients of these BDSDEs depend on the future value of the solution (Y,Z). We obtain the existence and uniqueness theorem and a comparison theorem for the solutions of these equations. Besides, as an application, we also establish a duality between the anticipated BDSDEs and the delayed doubly stochastic differential equations (delayed DSDEs).

1. Introduction

Backward stochastic differential equation (BSDE) was considered the general form the first time by Pardoux and Peng [10] in 1990. In the last twenty years, the theory of BSDEs has been studied with great interest due to its applications in the pricing/hedging problem (see e.g. [4,5]), in the stochastic control and game theory (see e.g. [5,6]), and in the theory of partial differential equations (see e.g. [2,3,11]).

In order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs), Pardoux and Peng [12] first studied the backward doubly stochastic differential equations (BDSDEs) of the general form

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{1.1}$$

where the integral with respect to $\{B_t\}$ is a "backward Itô integral", and the integral with respect to $\{W_t\}$ is a standard forward integral. Note that these two types of integrals are particular cases of the Itô-Skorohod integral, see Nualart and Pardoux [9]. Pardoux and Peng [12] proved that under Lipschitz condition on the coefficients, BDSDE (1.1) has a unique solution. Since then, the theory of BDSDEs has been developed rapidly by many researchers. Bally and Matoussi [1] gave the probabilistic representation of the solutions in Sobolev space of semilinear SPDEs in terms of BDSDEs. Matoussi and Scheutzow [8] studied BDSDEs and their applications in SPDEs. Shi et al. [14] proved a comparison theorem for BDSDEs with Lipschitz condition on the coefficients. Lin [7] obtained a generalized comparison theorem and a generalized existence theorem of BDSDEs.

On the other hand, recently, Peng and Yang [13] (see also [16]) introduced the socalled anticipated BSDEs (ABSDEs) of the following form:

$$\begin{cases} -dY_t = f(t,Y_t,Z_t,Y_{t+\delta(t)},Z_{t+\zeta(t)})dt - Z_t dW_t, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T,T+K]; \\ Z_t = \eta_t, & t \in [T,T+K], \end{cases}$$

where $\delta(\cdot) : [0,T] \to \mathbb{R}^+ \setminus \{0\}$ and $\zeta(\cdot) : [0,T] \to \mathbb{R}^+ \setminus \{0\}$ are continuous functions satisfying

E-mail address: xmxu@njnu.edu.cn

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(a1) there exists a constant $K \geqslant 0$ such that for each $t \in [0,T]$,

$$t+\delta(t) \leqslant T+K, \quad t+\zeta(t) \leqslant T+K;$$

(a2) there exists a constant $M \geqslant 0$ such that for each $t \in [0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s+\delta(s))ds \leqslant M\int_t^{T+K} g(s)ds, \quad \int_t^T g(s+\zeta(s))ds \leqslant M\int_t^{T+K} g(s)ds.$$

Peng and Yang [13] proved the existence and uniqueness of the solution to the above equation, and studied the duality between anticipated BSDEs and delayed SDEs.

In this paper, we are interested in the following BDSDEs with coefficients depending on the future value of the solution (Y,Z):

$$\begin{cases} -dY_t = & f(t,Y_t,Z_t,Y_{t+\delta(t)},Z_{t+\zeta(t)})dt \\ & +g(t,Y_t,Z_t,Y_{t+\delta(t)},Z_{t+\zeta(t)})d\overleftarrow{B}_t - Z_t dW_t, \quad t\in[0,T]; \\ Y_t = \xi_t, \quad t\in[T,T+K]; \\ Z_t = \eta_t, \quad t\in[T,T+K], \end{cases} \tag{1.2}$$

where $\delta > 0$ and $\zeta > 0$ satisfy (a1) and (a2).

We prove that under proper assumptions, the solution of the above anticipated BDSDE (ABDSDE) exists uniquely, and a comparison theorem is given for the 1-dimensional anticipated BDSDEs. It may be mentioned here that, to deal with (1.2), the most important thing for us is to establish the similar conclusions as in Ref. [12,14] for BDSDE (1.1) with ξ belonging to a larger space. Besides, as an application, we study a duality between the anticipated BDSDE and delayed DSDE.

The paper is organized as follows: in Section 2, we make some preliminaries. In Section 3, we mainly study the existence and uniqueness of the solutions of anticipated BDSDEs, and in Section 4, a comparison result is given. As an application, in Section 5, we establish a duality between an anticipated BDSDE and a delayed DSDE. Finally in Section 6, the conclusion and future work are presented.

2. Preliminaries

Let $T > 0$ be fixed throughout this paper. Let $\{W_t\}_{t\in[0,T]}$ and $\{B_t\}_{t\in[0,T]}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^l$, defined on a probability space $(\Omega,\mathcal{F},P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. We define

$$\mathcal{F}_t := \mathcal{F}^W_{0,t} \vee \mathcal{F}^B_{t,T}, \quad t\in[0,T]; \quad \mathcal{G}_s := \mathcal{F}^W_{0,s} \vee \mathcal{F}^B_{s,T+K}, \quad s\in[0,T+K],$$

where for any processes $\{\varphi_t\}$, $\mathcal{F}^\varphi_{s,t} = \sigma\{\varphi_r - \varphi_s, s\leqslant r\leqslant t\} \vee \mathcal{N}$. We will use the following notations:

- $L^2(\mathcal{G}_T;\mathbb{R}^m) := \{\xi\in\mathbb{R}^m \mid \xi$ is a $\mathcal{G}_T$-measurable random variable such that $E|\xi|^2 < +\infty\}$;
- $L^2_{\mathcal{G}}(0,T;\mathbb{R}^m) := \{\varphi : \Omega\times[0,T]\to\mathbb{R}^m \mid \varphi$ is a $\mathcal{G}_t$-progressively measurable process such that $E\int_0^T|\varphi_t|^2dt < +\infty\}$;
- $S^2_{\mathcal{G}}(0,T;\mathbb{R}^m) := \{\varphi : \Omega\times[0,T]\to\mathbb{R}^m | \varphi$ is a continuous and $\mathcal{G}_t$-progressively measurable process such that $E[\sup_{0\leqslant t\leqslant T}|\varphi_t|^2] < +\infty\}$.

Remark 2.1. It should be mentioned here that, the existing result about BDSDEs are established almost under the condition that the terminal value ξ is $\mathcal{F}_T$-measurable (see [12,14], etc.). In this paper, we will first treat the case when ξ is $\mathcal{G}_T$-measurable.

For each $t\in[0,T]$, let

$$f(t,\cdot,\cdot,\cdot,\cdot) : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^m)\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^{m\times d}) \to L^2(\mathcal{G}_t;\mathbb{R}^m),$$

$$g(t,\cdot,\cdot,\cdot,\cdot) : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^m)\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^{m\times d}) \to L^2(\mathcal{G}_t;\mathbb{R}^{m\times l}).$$

We make the following hypotheses:

(**H1**) There exists a constant $c > 0$ such that for any $r,\bar{r}\in[t,T+K], (t,y,z,\theta,\phi),(t,y',z',\theta',\phi')\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^m)\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^{m\times d})$,

$$|f(t,y,z,\theta_r,\phi_{\bar{r}}) - f(t,y',z',\theta'_r,\phi'_{\bar{r}})|^2 \leqslant c(|y-y'|^2 + |z-z'|^2 + E^{\mathcal{F}_t}[|\theta_r-\theta'_r|^2 + |\phi_{\bar{r}}-\phi'_{\bar{r}}|^2]).$$

(**H2**) $E\left[\int_0^T|f(s,0,0,0,0)|^2ds\right] < +\infty$.

(**H3**) There exist constants $c > 0, 0 < \alpha_1 < 1, 0 \leqslant \alpha_2 < \frac{1}{M}$, satisfying $0 < \alpha_1 + \alpha_2 M < 1$, such that for any $r,\bar{r}\in[t,T+K], (t,y,z,\theta,\phi),(t,y',z',\theta',\phi')\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^m)\times L^2_{\mathcal{G}}(t,T+K;\mathbb{R}^{m\times d})$,