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A recovery-based error estimate for nonconforming finite volume methods of interface problems

Lin Mu [a,*], Rabeea Jari [b]

[a] Department of Mathematics, Michigan State University, East Lansing, MI 48823, United States
[b] Mathematics Department, Thi-Qar University, Thi-Qar, Iraq

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ABSTRACT

In Cai and Zhang (2009, 2010) [12,13], they introduced the recovery-based a posteriori error estimator for conforming, mixed, and nonconforming finite element methods of interface problems. In this paper, we extend the idea to present a recovery-based a posterior error estimator for finite volume methods which employ the nonconforming linear trial functions to approximate elliptic interface problems. The method recovers the flux and gradient in $H(div)$ and $H(curl)$ conforming finite element spaces with a weighted L^2 projection, respectively. The reliability and efficiency bounds are established. Numerical experiments are given to support the conclusions.

1. Introduction

The finite volume method (FVM) is a discretization technique for solving partial differential equations (PDEs). Due to the property of local conservation of the interested quantity, FVM is widely used in many fields of engineering such as computational fluid dynamics.

An a posteriori error estimate is an important procedure for adaptivity methods. It uses numerical solution and several known data to estimate the error between the exact solution and the numerical solution. By the usage of computable quantities which can reflect the quality of numerical methods, it can be used as an indicator to refine the mesh where has large error, and thus reduce the computational error efficiently. There are two kinds of a posteriori error estimators: one is residual-based error estimator (see [10,28,35,4,8,20,7,15,6]) and the other is recovery-based error estimator (see [21,29,27]). The recovery-based methods have been widely used in engineering, and studied by many researchers, (for example [2,3,9,30,33,34,36,18]), because of their many appealing properties.

The purpose of this paper is to derive a recovery-based a posteriori error estimator for finite volume methods of solving interface problems. Interface problems with discontinuous coefficients are the popular examples in material sciences and fluid dynamics, for example two distinct materials or fluids with different conductivities, densities or diffusions. Cai and Zhang introduced the recovery-based a posteriori error estimate for interface problems by conforming finite element methods [12] and mixed and nonconforming finite element methods [13]. The same problem is studied for conforming finite volume methods in [23]. Also it is studied in [12,13] for finite element methods. In this paper, we will extend the idea to nonconforming trial functions for finite volume methods. The error estimator is derived by using the flux and gradient, which are recovered in $H(div)$ and $H(curl)$ conforming finite element spaces, respectively. With an assumption on the monotonicity distribution of the coefficients, both the reliability and efficiency constants are proved to be independent of the size

* Corresponding author.
E-mail addresses: linmu@math.msu.edu, lxmu@ualr.edu (L. Mu), rhjari@ualr.edu (R. Jari).

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of jumps, the new estimators are robust with respect to the diffusion coefficients. The numerical experiments are provided on classical problems.

This paper is organized as follows. In Section 2, function spaces and preliminaries are introduced. In Section 3, the nonconforming linear finite volume approximation for interface problems are derived. In Section 4, we present the recovery procedure and the resulting recovery-based a posteriori error estimator. In Section 5, the a posteriori error analysis is derived. Finally, in Section 6, we give numerical results to support our analysis.

2. Function spaces and preliminaries

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. For a two dimensional vector-valued function $\boldsymbol{\tau} = (\tau_1, \tau_2)^t$, and a scalar-valued function v, the divergence, curl, and vector curl operators are defined as following

$$\nabla \cdot \boldsymbol{\tau} := \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2}, \quad \nabla \times \boldsymbol{\tau} := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}, \text{ and } \nabla^{\perp} v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1} \right)^t.$$

The standard notations and definitions will be adopted for the Sobolev spaces (see [1]): $H^s(K),\ H^s(\partial K)\ (s \geqslant 0)$ and inner products $(\cdot,\cdot)_{s,K},\ (\cdot,\cdot)_{s,\partial K}$, norms $\|\cdot\|_{s,K}, \|\cdot\|_{s,\partial K}$ and semi-norms $|\cdot|_{s,K}$, $(s \geqslant 0)$, where $K \subset \Omega$. If $s = 0,\ H^0(K) = L^2(K)$, in which case the inner product is denoted by $(\cdot,\cdot)_K$. If $K = \Omega$, we drop K.

Moreover, the following Hilbert spaces will be used:

$$H(div;\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 : \nabla \cdot \boldsymbol{\tau} \in L^2(\Omega)\}$$

and

$$H(curl;\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 : \nabla \times \boldsymbol{\tau} \in L^2(\Omega)\},$$

equipped with the norms

$$\|\boldsymbol{\tau}\|_{H(div;\Omega)} := \left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\nabla \cdot \boldsymbol{\tau}\|_{0,\Omega}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{H(curl;\Omega)} := \left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\nabla \times \boldsymbol{\tau}\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

Denote the subspace by

$$H_0(curl;\Omega) = \{\boldsymbol{\tau} \in H(curl;\Omega) : \boldsymbol{\tau} \cdot \mathbf{t}|_{\partial\Omega} = 0\},$$

where $\mathbf{t}$ is the unite vector clockwise tangent to the boundary $\partial\Omega$.

Finally, the following formula of integration by parts will be used

$$(\nabla \times \boldsymbol{\tau}, v) + (\boldsymbol{\tau}, \nabla^{\perp} v) = \int_{\partial\Omega} (\boldsymbol{\tau} \cdot \mathbf{t}) v ds, \tag{2.1}$$

for all $\boldsymbol{\tau} \in H(curl;\Omega)$ and all $v \in H^1(\Omega)$.

3. Interface problems and finite volume approximation

Consider the following interface problem

$$-\nabla \cdot (k(\mathbf{x})\nabla u) = f \quad \text{in} \quad \Omega, \tag{3.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

where f is a given scalar-valued function. Here $k(\mathbf{x}) \geqslant 0$ is piece-wise constant on polygonal sub-domains of $\Omega_i \subset \Omega$ $(i = 1, \ldots, n)$ and accepts possible large jumps across sub-domain boundaries (interfaces). Define

$$k_{\min} = \min_{1 \leqslant i \leqslant n} k_i \quad \text{and} \quad k_{\max} = \max_{1 \leqslant i \leqslant n} k_i.$$

In order to define the FVM, we need two different partitions of the domain Ω. One is called the primal partition, which is associated with the trial function space. The other one is a dual partition, which is associated with the test function space. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with $diam(K) \leqslant h,\ \forall K \in \mathcal{T}_h$. Furthermore, assume that the interfaces $F = \{\partial\Omega_i \cap \partial\Omega_j | i,\ j = 1, \ldots, n\}$ do not cut through element $K \in \mathcal{T}_h$. We obtain the dual mesh $\mathcal{T}_h^*$ of $\mathcal{T}_h$ for the test function space as follows: each element in $\mathcal{T}_h^*$ is made up of two subtriangles which share a common edge (see Fig. 1). These subtriangles are formed by connecting the barycenter and the two corners of the triangles. In Fig. 1, the control volume for point P is the hull EB_1CB_2. Let $\mathcal{E}_h$ denote the union of the boundaries of the triangles $K \in \mathcal{T}_h,\ \mathcal{E}_h^0 := \mathcal{E}_h \setminus \partial\Omega$. Moreover, denote $k_K = k(\mathbf{x})|_K,\ \forall K \in \mathcal{T}_h$.

Let $P_l(K)$ be the space of polynomials of degree l on element K. Define conforming and nonconforming piecewise linear spaces by

$$V_c = \{v \in H_0^1(\Omega) : v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}$$