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On half-discrete Hilbert's inequality ☆

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ABSTRACT

In this paper we provide an application of the Euler–Maclaurin summation formula with the Bernoulli function for the proof of a strengthened version of the half-discrete Hilbert inequality with the best constant factor in terms of the Euler–Mascheroni constant. Some equivalent numerical representations, operator representations, two kinds of reverses as well as an extension in terms of parameters and the Beta function are also studied.

1. Introduction

Assuming that $p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ f(x), g(y) \geqslant 0,\ f \in L^p(\mathbb{R}_+),\ g \in L^q(\mathbb{R}_+)$,

$$||f||_p = \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} > 0,$$

$||g||_q > 0$, the following Hardy–Hilbert integral inequality (cf. [1]) holds

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1.1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible.

If $a_m,\ b_n \geqslant 0,\ a = \{a_m\}_{m=1}^\infty \in l^p,\ b = \{b_n\}_{n=1}^\infty \in l^q,\ ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0,\ ||b||_q > 0$, then we have the following discrete Hardy–Hilbert inequality

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{1.2}$$

with the same best constant factor $\pi/\sin(\pi/p)$.

Inequalities of the form (1.1) and (1.2) are important in analysis and its applications (cf. [1–3]).

In 1998, Yang [4] gave an extension of (1.1) for $p = q = 2$, by introducing an independent parameter $\lambda \in (0, 1]$. In order to generalize the results of [4], Yang in [5,6] gave some extensions of (1.1) and (1.2) as follows: If $\lambda_1 + \lambda_2 = \lambda \in \mathbb{R},\ k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$,

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$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbb{R}_+, \tag{1.3}$$

$$\phi(x) = x^{p(1-\lambda_1)-1}, \quad \psi(x) = x^{q(1-\lambda_2)-1},$$

$$f(x), \quad g(y) \geqslant 0, \quad f \in L_{p,\phi}(\mathbb{R}_+) = \left\{ f | ||f||_{p,\phi} := \left\{ \int_0^\infty \phi(x)|f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}$$

and

$$g \in L_{q,\psi}(\mathbb{R}_+), \quad ||f||_{p,\phi}, \quad ||g||_{q,\psi} > 0,$$

then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dx\,dy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}k_\lambda(x,y)y^{\lambda_2-1}$ is strictly decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, \ b_n \geqslant 0$,

$$a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} := \left\{ a | ||a||_{p,\phi} := \left\{ \sum\nolimits_{n=1}^\infty \phi(n)|a_n|^p \right\}^{\frac{1}{p}} < \infty \right\}$$

and

$$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, \quad ||a||_{p,\phi}, \quad ||b||_{q,\psi} > 0,$$

we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.5}$$

with the same best possible constant factor $k(\lambda_1)$.

Clearly, for $\lambda = 1, \ k_1(x,y) = 1/(x+y), \ \lambda_1 = 1/q, \ \lambda_2 = 1/p$, (1.4) reduces to (1.1), while (1.5) reduces to (1.2). Some other results including the reverse Hilbert-type inequalities and the operator expressions are provided by [7–23,59].

Hardy, Littlewood and Pólya presented in Theorem 351 of [1] a few results on half-discrete Hilbert-type inequalities with non-homogeneous kernels. However, they did not prove that the constant factors in the inequalities are the best possible. Yang in [24] proved the following result on the non-homogeneous Hilbert kernel $1/(1+nx)^\lambda, 0 < \lambda \leqslant 2$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{(1+nx)^\lambda} dx < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=1}^\infty n^{\frac{\lambda}{2}-1} a_n^2 \right\}^{\frac{1}{2}} \left\{ \int_0^\infty x^{\frac{\lambda}{2}-1} f^2(x) dx \right\}^{\frac{1}{2}}, \tag{1.6}$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible and $B(u, v)$ stands for the Beta function. Some other results related to half-discrete Hilbert-type inequalities are presented in [25–45].

In this paper, by the use of the method of weight functions and techniques of Real Analysis, we apply the Euler–Maclaurin summation formula with the Bernoulli function to give a strengthened version of the half-discrete Hilbert inequality with the best constant factor involving the Euler–Mascheroni constant, as follows:

$$\begin{aligned} I &:= \int_1^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{n+x} dx = \sum_{n=1}^\infty a_n \int_1^\infty \frac{f(x)}{n+x} dx \\ &< \left\{ \sum_{n=1}^\infty \left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{\ln 2}{n^{1/r}} \right] n^{\frac{p}{r}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \int_1^\infty \left[\frac{\pi}{\sin\left(\frac{\pi}{r}\right)} - \frac{1-\gamma}{x^{1/s}} \right] x^{\frac{q}{s}-1} f^q(x) dx \right\}^{\frac{1}{q}} \end{aligned} \tag{1.7}$$

where γ is the Euler–Mascheroni constant, $p, r > 1$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$.

The equivalent forms, the operator expressions, two kinds of reverses as well as the extension with parameters and the Beta function are also considered.

Remark 1.1. The Euler–Mascheroni constant $\gamma = \gamma_0$ is also called Stieltjes constant of order 0, which is the first constant of the Laurent series of the Riemann zeta function $\zeta(s)$ in the isolated singular point $s_0 = 1$, as presented in the following formula (cf. [6]):

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^\infty (-1)^n \frac{\gamma_n}{n!}(s-1)^n, \quad 0 < |s-1| < \infty. \tag{1.8}$$

The Riemann zeta function, the Gamma function, the Beta function, etc., are of great importance in Number Theory and its applications. For the modern development of Analytic Number Theory, refer to [46–54].