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A modification of the Shishkin discretization mesh for one-dimensional reaction–diffusion problems

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ABSTRACT

In this paper we consider a modification of the Shishkin discretization mesh designed for the numerical solution of one-dimensional singularly perturbed reaction–diffusion problems. The modification consists of a slightly different choice of the transition points between the fine and coarse parts of the mesh. We prove that this change does not affect the order of convergence of the numerical solution obtained by using the central finite-difference scheme. However, due to a better layer-resolving mesh, numerical results show an improvement in the accuracy of the computed solution when compared to the results on the standard Shishkin mesh.

1. Introduction

It is well known that numerical methods for singularly perturbed problems have to be very carefully created. This is because the solution of a singularly perturbed problem usually has boundary or interior layers, which are regions where the solution changes abruptly. Surveys of numerical methods for these problems can be found in [1–5]. One way to construct a numerical method which is efficient and uniform in the perturbation parameter is to use an appropriate finite-difference scheme on the layer-adapted Shishkin mesh. The Shishkin mesh is discussed at length in [1,2] and many numerical methods are known to be stable and accurate on it [3–5].

In this paper we consider a one-dimensional reaction–diffusion problem, the solution of which has two boundary layers in general. For this problem, the Shishkin mesh is divided into two fine parts in the layers and the coarse one outside the layers. The points at which the mesh step size changes are called transition points. The influence of the choice of the transition point and the complete mechanism of the Shishkin mesh are explained in details in [6]. Improvements and generalizations of the Shishkin mesh can be found in [3,7,8].

The aim of this paper is to improve the performance of the Shishkin mesh by redefining the transition points so that they more closely correspond to the boundary-layer functions which describe how the solution behaves in the layers. This results in a better layer-resolving mesh because the transition points are moved closer to the end points where the layers occur. Thus we get a higher density of mesh points in the layers, because of which we can expect more accurate numerical results. The motivation for this work comes from [9], where a modified transition point is used in numerical experiments, although the theoretical analysis is carried out for the standard transition point. We prove here that the use of modified transition points is theoretically justified. However, the problem we consider is different from the quasilinear convection–diffusion problem of [9], which we shall deal with elsewhere. Our proof is based on a solution decomposition which is analogous

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to that of [10] for the linear convection–diffusion problem. We also show that this decomposition can be used to improve the Shishkin decomposition of the solution which is often used in the numerical analysis of linear singular perturbation problems, see [1–4,6].

The one-dimensional problem we analyze here can be considered as a model problem. It is to be expected that our results can be extended to two-dimensional reaction–diffusion problems. Results of the same type for nonlinear singular perturbation problems in one dimension are currently under preparation.

The outline of the paper is as follows. In Section 2 we analyze the continuous solution. The discretization scheme and the corresponding Green's function are introduced in Section 3. This is followed by the definition of the modified Shishkin mesh and the proof of the convergence result. In Section 4 we prove that the piecewise linear interpolation of the numerical solution retains the accuracy of the numerical method. Section 5 provides results of numerical experiments, which show improvements in the computed solution when compared to the results on the standard Shishkin mesh.

2. The continuous problem

We consider the problem

$$\begin{aligned} &\Lambda y(x) := -\varepsilon^2 y''(x) + b^2(x)y(x) = f(x), \quad x \in (0,1), \\ &y(0) = \alpha_0, \quad y(1) = \alpha_1, \end{aligned} \tag{1}$$

where ε is a small positive perturbation parameter, $b, f \in C^4[0,1]$, and

$$b(x) > \beta > 0, \quad x \in [0,1].$$

It is well known that this problem has a unique solution $y \in C^6[0,1]$, for which the following estimates hold true (see [3,11] for instance):

$$|y^{(i)}(x)| \leqslant M\left[1 + \varepsilon^{-i} v_0(x) + \varepsilon^{-i} v_1(x)\right], \quad x \in [0,1], \quad i = 0,1,\ldots,4, \tag{2}$$

where

$$v_0(x) = e^{-\beta x/\varepsilon} \quad \text{and} \quad v_1(x) = e^{-\beta(1-x)/\varepsilon}.$$

Above and throughout the paper, M denotes any positive constant which is independent of ε. Some particular constants of this kind will be subscripted. As can be seen from (2), the solution exhibits in general two layers of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ at both endpoints of the domain.

The estimates (2) follow from an appropriate solution decomposition given in [3, Theorem 3.35] and [11] (the technique used there allows for $\beta = \min_{x\in[0,1]} b(x)$, but we need $b > \beta$ in the proof of Lemma 3, which is used in the proof of Theorem 2). This decomposition is of Shishkin type [1–4,6] because it decomposes the solution into the smooth and boundary-layer parts. We present below the version from [11]:

$$\begin{aligned} &y(x) = s(x) + v(x), \\ &|s^{(i)}(x)| \leqslant M, \quad |v^{(i)}(x)| \leqslant M\varepsilon^{-i}[v_0(x) + v_1(x)], \\ &x \in [0,1], \quad i = 0,1,\ldots,4. \end{aligned} \tag{3}$$

For the construction of the function s, see [11] since the details are not important here. As for v, it solves the problem

$$\Lambda v(x) = 0, \quad x \in (0,1), \quad v(0) = \alpha_0 - s(0), \quad v(1) = \alpha_1 - s(1). \tag{4}$$

Both (2) and (3) can be improved using the technique from [10]. This technique, originally devised for convection–diffusion problems, has been adapted to reaction–diffusion problems (1) in [12]. The result from [12] is given in the following lemma in a slightly stronger form. We use

$$w_0(x) = e^{-b(0)x/\varepsilon}, \quad w_1(x) = e^{-b(1)(1-x)/\varepsilon}.$$

Lemma 1. *For the solution y of (1) we have*

$$y(x) = pw_0(x) + z_0(x), \quad 0 \leqslant x \leqslant \frac{1}{2},$$

$$y(x) = qw_1(x) + z_1(x), \quad \frac{1}{2} \leqslant x \leqslant 1,$$

where p and q are constants satisfying $|p|, |q| \leqslant M$, and

$$|z_j^{(i)}(x)| \leqslant M\left[1 + \varepsilon^{1-i} v_j(x)\right], \quad j = 0,1, \quad i = 0,1,\ldots,4.$$