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On explicit formulas for the principal matrix logarithm

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ABSTRACT

We describe a method for evaluating both the Fibonacci-Hörner and the polynomial decomposition of the principal matrix logarithm, with a view to solve the lifting problem of its explicit computation. The Binet formula for linear recursive sequences serves as a triggering factor for giving the exact formula. We supply some illustrative examples.

1. Introduction

The logarithms of a matrix appear in various fields of mathematics, applied sciences and engineering. Many methods and algorithms are expanded in producing their representations (see [7,9–11,15,18] and references therein). In contrast with the matrix exponential, the definition of the matrix logarithms reveals some significant difficulties. Indeed, for a matrix B in the algebra of square matrices, $M_d(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), the problem consists in finding a matrix $X \in M_d(\mathbb{K})$ satisfying the matrix equation $e^X = B$. Any solution of this equation, denoted by $X = log(B)$, is called *logarithm of B*. A matrix $B \in M_d(\mathbb{K})$ has a logarithm (not necessary real) if and only if B is nonsingular. Nevertheless, the equation $e^X = B$ may have infinitely many solutions; see e.g. [7,9–11]. By way of contrast, if B has no eigenvalues on the closed negative real axis, among its infinitely many logarithms, there exists a unique logarithm X called the *principal logarithm* of B and denoted by $X = Log(B)$. This unique matrix logarithm has all its eigenvalues into the horizontal strip determined by the condition $\{\lambda_i(X) \in \mathbb{C} : |Im(\lambda_i(X))| < \pi\}$ (see [7,10,11,15]). Meanwhile, the simplest manner to define the principal matrix logarithm is a power series. This definition is based on the fact that the function $log(z) = Log|z| + iarg(z)$ is analytical in its principal branch, $|z| > 0$ and $|Im(log(z))| < \pi$. The series $Log(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}$, in the unit disk centered at $z_0 = 1$, can be applied to define the principal matrix logarithm as $Log(B) = -\sum_{n=1}^{\infty} \frac{(I_d-B)^n}{n}$, for $\|I_d - B\| < 1$.

The computation of the principal matrix logarithm stills an exciting area, and the main purpose here is to provide methods in order to compute exactly the principal logarithm of a matrix $B \in GL(d, \mathbb{C})$ under some general conditions, where $GL(d, \mathbb{C})$ represents the group of invertible matrices of order d. The quote part of our study is exhibited in the usage of techniques involving elementary properties of linear Fibonacci sequences and their Binet formula, which allows us to establish a simpler and explicit formula of the principal matrix logarithm. Our methods are based on the knowledge of an annihilator polynomial $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ of $A = I_d - B$, and the Fibonacci-Hörner decomposition of its power matrices $A^n, n \in \mathbb{N}$. Our development is released from the general properties of the linear recurrence relations and the Cayley-Hamilton Theorem (Section 2). We introduce the Binet formula for the solutions of linear recurrence relations to obtain an analytical and exact formula for the principal logarithm of a matrix in its Fibonacci-Hörner basis (Section 3). Over and above, illustrative examples, remarks, and comparisons with other known representations for the principal matrix logarithm are also provided.

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2. Decompositions of the principal matrix logarithm via recursive relations

2.1. Fibonacci-Hörner decomposition and recursiveness

Let $A \in M_d(\mathbb{C})$ and $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$ a polynomial such that $R(A) = \Theta_d$ (the zero matrix of order d), with $2 \leqslant r \leqslant d$ to avoid trivialities. Following [3,5,13], we have

$$A^n = u_n A_0 + u_{n-1} A_1 + \cdots + u_{n-r+1} A_{r-1}, \text{ for } n \geqslant 0, \tag{1}$$

where the sequence $\{u_n\}_{n \geqslant -r+1}$ is defined by $u_0 = 1$, $u_{-r+1} = \cdots = u_{-1} = 0$, and $u_m = \sum_{k_0+2k_1+\cdots+rk_{r-1}=m} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\cdots k_{r-1}!} a_0^{k_0} \cdots a_{r-1}^{k_{r-1}}$, for $m \geqslant 1$, represents its combinatorial expression. Moreover, as shown in [3,4,13], the sequence $\{u_n\}_{n \geqslant -r+1}$ satisfies the linear recursive relation of order r,

$$u_{n+1} = a_0 u_n + \cdots + a_{r-1} u_{n-r+1}, \text{ for } n \geqslant 0, \tag{2}$$

with constant coefficients $a_0, a_1, \cdots, a_{r-1}$ (see [2,8], for example). Sequences (2) are known in the literature as *r-generalized Fibonacci sequences*. The set of matrices $A_0 = I_d, A_1, \ldots, A_{r-1}$ from (1) is the so-called *Fibonacci-Hörner basis* associated to the matrix A,

$$A_0 = I_d\,, A_k = A^k - a_0 A^{k-1} - \ldots - a_{k-1} I_d, (\text{viz } A_k = A A_{k-1} - a_{k-1} I_d) \tag{3}$$

for $1 \leqslant k \leqslant r-1$ (see [4,13]). Expression (1) is the so-called *Fibonacci-Hörner decomposition* of the power matrix A^n (see [5]). Furthermore, the polynomial decomposition of A^n (for $n \geqslant r$) in its power basis $\{A^s\}_{0 \leqslant s \leqslant r-1}$, obtained by a straightforward computation, is

$$A^n = \sum_{p=0}^{r-1} \left(\sum_{j=0}^{p} a_{r-p+j-1} u_{n-r-j} \right) A^p, \tag{4}$$

(see Proposition 3.1 of [4] and Corollary 2.2 of [13]). For computing explicit representations of the Fibonacci-Hörner decomposition (1) and the polynomial decomposition (4) of the power matrices A^n, the expression of u_m ($m \geqslant 1$) are obtained recursively by using (2). However, we can also get these coefficients by giving (2) under a determinantal form. That is, for $m \geqslant 1$, the determinantal representation of u_m using a $m \times m$ upper Hessenberg matrix [12,18], is as follows

$$u_m = \begin{vmatrix} a_0 & a_1 & \ldots & a_{r-1} & 0 & \ldots & 0 \\ -1 & a_0 & \ldots & a_{r-2} & a_{r-1} & \ldots & 0 \\ \vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \ddots & \ldots & \ldots & \vdots \\ 0 & 0 & \ldots & \ldots & \ldots & a_0 & a_1 \\ 0 & 0 & \ldots & \ldots & \ldots & -1 & a_0 \end{vmatrix}. \tag{5}$$

Note that for $r+1 \leqslant m$, Expression (5) results from the determinant of a Toeplitz (r + 1)-banded matrix.

Remark 2.1. The powers of singular matrices $A \in M_d(\mathbb{C})$ are also included in the representation (1). For such matrices the polynomial $R(z)$ accomplishes $a_{r-1} = 0$. Its Fibonacci-Hörner decomposition is provided by considering in the relation (2) the zero coefficient a_{r-1}; see Example 3.2 for more details.

Although the following result is no longer used in the following, drawing one's inspiration from the representation (5) of u_m, we propose a new determinantal representation for the n-th power matrix A^n ($n \geqslant r$) in the power basis.

Proposition 2.2. *Under the preceding data we have,*

$$A^n = \sum_{k=0}^{r-1} u_{n-r+1}^{(k)} A^{r-1-k} = \sum_{k=0}^{r-1} u_{n-r+1}^{(r-1-k)} A^k, \tag{6}$$

where $u_{n-r+1}^{(k)}$, the determinant of a $(n-r+1) \times (n-r+1)$ matrix, is as the u_{n-r+1} given in Expression (5), but with the first row shifted k positions forward, [12]. Note that $u_{n-r+1}^{(0)} = u_{n-r+1}$ is as given in (5).

Proof. By mathematical induction. Since $R(A) = \Theta_d$, the result for A^r is trivial. We assume that A^{n-1} has the following representation in the power basis, $A^{n-1} = \sum_{k=0}^{r-1} u_{n-r}^{(k)} A^{r-1-k}$. Therefore, using the induction, we get

$$A^n = u_{n-r}^{(0)} A^r + \sum_{k=1}^{r-1} u_{n-r}^{(k)} A^{r-k} = u_{n-r}^{(0)} \sum_{k=0}^{r-1} a_k A^{r-1-k} + \sum_{k=0}^{r-2} u_{n-r}^{(k+1)} A^{r-1-k}.$$