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Exponential attractors for the strongly damped wave equation ☆

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ABSTRACT

In this paper, we study the longtime dynamics to a strongly damped wave equation. First, in the critical case, we prove by the *l*-trajectories method that the related solution semigroup has the exponential attractor. Second, in the subcritical case, we give an explicit upper bound of the fractal dimension of the exponential attractor in terms of the data of the equation.

1. Introduction

We consider the following strongly damped wave equation on a bounded domain $\Omega \subset \mathbf{R}^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad u_t(0) = u_1, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)$, $\varphi \in C^2(\mathbf{R})$ with $\varphi(0) = 0$,

$$\liminf_{|r|\to\infty} \varphi'(r) > -\lambda_1, \quad r \in \mathbf{R}, \tag{1.2}$$

where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, and φ is of critical growth

$$\varphi''(r) \leqslant C(1 + |r|^3), \quad r \in \mathbf{R}. \tag{1.3}$$

The longtime dynamics of problem (1.1) has been investigated quite extensively by several authors (see [1–4] and references therein) in the subcritical case.

For the critical case (1.3), Carvalho and Cholewa ([5]) achieved the existence of the global attractor of the dynamical system associated with problem (1.1). The additional regularity of the global attractor was also obtained in [6,7]. Although the global attractor is a powerful tool to investigate the longtime behavior of the dissipative dynamical system, it is difficult in general to exhibit an actual control of the convergence rate of the trajectories tending to the attractor. In order to solve the question, the concept of the exponential attractor was introduced. But for the critical case, does the solution semigroup $S(t)$ related to problem (1.1) has any exponential attractor? The question is not fully discussed. The first part of the paper aims to establish an exponential attractor for the critical case.

For the subcritical case (i.e. the growth order of φ is less than 5), by using the bootstrap argument, Pata and Squassina ([3]) established an exponential attractor of optimal regularity for the strongly damped wave equation with a damping parameter ω,

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$$u_{tt} - \omega\Delta u_t - \Delta u + \varphi(u) = f.$$

The second part of the paper is to strengthen the results in [3]. By using the method of l-trajectories, we estimate the upper bound of the fractal dimension of an exponential attractor which tends to 0 as $\omega \to \infty$, and tends to ∞ as $\omega \to 0$. The estimation improves in some extent the results in [8] and [3, Remark 10], where the fractal dimension of the attractor remains bounded as $\omega \to \infty$.

2. Preliminary

We denote the inner product and the norm on $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $A = -\Delta$ with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. We consider a family of Hilbert spaces $H_s = \mathcal{D}(A^{s/2}),\ s \in\in R$, whose inner products and norms are given by

$$(\cdot,\cdot)_{H_s} = (A^{s/2}\cdot, A^{s/2}\cdot) \quad \text{and} \quad \|\cdot\|_{H_s} = \|A^{s/2}\cdot\|.$$

Obviously,

$$H_0 = L^2(\Omega), \quad H_1 = H_0^1(\Omega), \quad H_2 = H^2(\Omega) \cap H_0^1(\Omega)$$

and

$$H_s \hookrightarrow\hookrightarrow H_r, \quad \text{and} \quad \|v\|_{H_r} \leqslant \lambda_1^{(r-s)/2}\|v\|_{H_s}, \quad \forall s > r,$$

where λ_1 is the first eigenvalue of A.

We introduce the Hilbert spaces

$$\mathcal{H} = H_1 \times H_0, \quad \tilde{\mathcal{H}} = H_2 \times H_1$$

endowed with the usual inner products and norms.

We consider the following abstract Cauchy problem which is equivalent to problem (1.1),

$$\begin{cases} u_{tt} + Au_t + Au + \varphi(u) = f, \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \tag{2.1}$$

Throughout the paper, C stands for a generic positive constant and Q stands for a generic positive increasing function, respectively. For any function $u(t)$, we write for short $\xi_u(t) = (u(t), \partial_t u(t))$.

Under assumptions (1.2) and (1.3), the solution operator $S(t)$ is well-defined in the phase space $\mathcal{H}$ and it defines a strongly continuous semigroup on $\mathcal{H}$ (see [1,5]), and the following results hold:

Lemma 2.1 [3]. *Given any $R > 0$ and any $\xi_u(0),\ \xi_v(0) \in \mathcal{H}$, with $\|\xi_u(0)\|_{\mathcal{H}},\ \|\xi_v(0)\|_{\mathcal{H}} \leqslant R$, there holds*

$$\|S(t)\xi_u(0) - S(t)\xi_v(0)\|_{\mathcal{H}} \leqslant e^{\kappa_1 t}\|\xi_u(0) - \xi_v(0)\|_{\mathcal{H}}, \quad \forall t \in R^+, \tag{2.2}$$

for some $\kappa_1 = \kappa_1(R)$.

Lemma 2.2 [7]. *The inequality*

$$\|\xi_u(t)\|_{\mathcal{H}}^2 + \int_t^\infty \|A^{\frac{1}{2}}u_t(\tau)\|^2 d\tau \leqslant Q(\|\xi_u(0)\|_{\mathcal{H}})e^{-\kappa_2 t} + Q(\|f\|) \tag{2.3}$$

holds for every $t \geqslant 0$ and some $\kappa_2 > 0$, where $\xi_u(t) = S(t)\xi_u(0)$.

Lemma 2.3 [6]. *There exists a bounded set $\mathbb{B}_1 \subset \tilde{\mathcal{H}},\ \kappa_3 > 0$ such that*

$$\mathrm{dist}_{\mathcal{H}}(S(t)B, \mathbb{B}_1) \leqslant Q(\|B\|_{\mathcal{H}})e^{-\kappa_3 t}, \quad \forall t \in \mathbf{R}^+,$$

for every bounded set $B \subset \mathcal{H}$.

Remark 2.1. By Lemmas 2.2 and 2.3, we learn at once that there exists a bounded absorbing set $\mathbb{B}_0 \subset \mathcal{H}$ for the solution semigroup $S(t)$ on $\mathcal{H}$ such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_0, \mathbb{B}_1) \leqslant Q(\|\mathbb{B}_0\|_{\mathcal{H}})e^{-\kappa_4 t}, \quad \kappa_4 > 0,\ \forall t \in \mathbf{R}^+. \tag{2.4}$$

3. Exponential attractor for the critical case

Theorem 3.1. *Let $f \in L^2(\Omega),\ \varphi \in C^2(\mathbf{R})$ with $\varphi(0) = 0$, (1.2) and (1.3) hold. Then the semigroup $S(t)$ acting on $\mathcal{H}$ possesses an exponential attractor $\mathcal{M}$ in the following sense:*