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Constant mean curvature surfaces with boundary on a sphere

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ABSTRACT

In this article we study the shape of a compact surface of constant mean curvature of Euclidean space whose boundary is contained in a round sphere. We consider the case that the boundary is prescribed or that the surface meets the sphere with a constant angle. We study under what geometric conditions the surface must be spherical. Our results apply in many scenarios in physics where in absence of gravity a liquid drop is deposited on a round solid ball and the air–liquid interface is a critical point for area under all variations that preserve the enclosed volume.

1. Introduction

Surfaces with constant mean curvature, abbreviated by cmc surfaces, are mathematical models of soap films and soap bubbles and, in general, of interfaces and capillary surfaces. Under conditions of homogeneity and in absence of gravity, an interface attains a state of physical equilibrium when minimizes the surface area enclosing a fixed volume, or at least, when it is a critical point for the area under deformations that preserve the volume. The literature on capillarity is extensive and we refer the classical text of Finn [7]; applications in physics and technological processes appear in [1,8,13]. We will study two physical phenomena. First, a liquid drop W deposited on a surface Σ in such way that the wetted region by W on Σ is a prescribed domain $\Omega\subset\Sigma$. Then the air–liquid interface $S=\partial W-\Omega$ is a compact cmc surface whose boundary curve ∂S is prescribed to be $\partial\Omega$. The second example appears in contexts of wetting and capillarity. Consider a liquid drop W deposited on a given support Σ and such that W can freely move along Σ. Here the curve ∂S remains on Σ but now is not prescribed. In equilibrium, the interface S is a cmc surface and S meets Σ with a constant contact angle γ, where γ depends on the materials.

Both contexts correspond with two mathematical problems. Denote by S a compact smooth surface with boundary ∂S. The first problem is as follows. Given a closed smooth curve $\Gamma\subset\mathbb{R}^3$, study the shape of a compact cmc surface S whose boundary is $\partial S=\Gamma$. For example, we ask whether the geometry of Γ imposes restrictions to the possible configurations of S, such as, if the symmetries of Γ are inherited by S. To be precise, suppose that $\Phi:\mathbb{R}^3\to\mathbb{R}^3$ is an isometry such that $\Phi(\Gamma)=\Gamma$ and let S be a cmc surface spanning Γ. Then we ask if $\Phi(S)=S$. The simplest case of boundary is a circle contained in a plane Σ. This curve is invariant by rotations with respect to the straight line L orthogonal to Σ through the center of Γ. If Γ is a circle of radius $r>0$ and $H\neq 0$, there exist two spherical caps (the large and the small one) bounded by Γ and with radius $1/|H|$, $0<|H|r\leqslant 1$ (if $|H|=1/r$, then both caps are hemispheres). Also, the planar disk bounded by Γ is a cmc surface

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with $H = 0$. All these examples are rotational surfaces where the axis of revolution is L. However, it should be noted that Kapouleas found non-rotational compact cmc surfaces bounded by a circle [10].

When the surface is embedded we can apply the so-called Alexandrov reflection method (or the method of moving planes) based on the maximum principle for elliptic partial differential equations of second order, which consists in a process of reflection about planes, using the very surface as a barrier [2]. Thanks to this technique, given a circle Γ contained in a plane Σ, if S is a compact embedded cmc surface with $\partial S = \Gamma$ and S lies on one side of Σ, then S is a spherical cap. Recall that Kaopuleass examples are non-embedded surfaces and lies on one side of the boundary plane. The lack of examples provides us evidence supporting the next.

Conjecture. A compact embedded cmc surface in $\mathbb{R}^3$ spanning a circle is a spherical cap or a planar disk.

Thus, the conjecture reduces to the question under what conditions such a surface lies on one side of Σ. Some partial answers have been obtained in [4,12,16]. However it is not known if there exists a cmc surface spanning a circle as in Fig. 1, left.

The mathematical formulation of the second setting is the following. Consider a regular region $R \subset \mathbb{R}^3$ with $\Sigma = \partial R$. Let S be a compact surface S with $int(S) \subset int(R)$ and $\partial S \subset \Sigma$ separating a bounded domain $W \subset R$ with a prescribed volume. The domain W is bounded by S and by pieces of Σ. Let $\gamma \in [0, \pi]$. We seek a surface S which is critical for the energy functional $\mathrm{area}(S) - (\cos\gamma)\mathrm{area}(\partial W \cap \Sigma)$ in the space of compact surfaces with boundary contained in Σ and interior contained in $int(R)$ and preserving the volume of W. In such case, we say that S is a stationary surface. A stationary surface is characterized by the fact that its mean curvature H is constant and S meets Σ in a constant angle γ along ∂S.

In this article we shall consider both problems when the supporting surface Σ is a sphere. First in Section 2 we study compact embedded cmc surfaces with prescribed boundary on a sphere Σ. We give results showing that the surface inherits some symmetries of its boundary, and we prove that the Conjecture is true in some special cases. See Fig. 1, right. In Section 3, we study stationary surfaces with boundary on a sphere. In physics, these configurations appear in the context of capillarity, for example, [6,11,17,19,25,27]. In the case that we study here, say, Σ is a sphere, a result of Taylor asserts that the boundary of a stationary surface is smooth because of Σ [24]. With the above notation, if R is the closed ball defined by Σ, there are examples of stationary surfaces intersecting Σ with a contact angle: besides the planar disks and spherical caps, whose boundary is a circle contained in Σ, there are pieces of rotational (non spherical) cmc surfaces whose boundary is formed by two coaxial circles [5]. Nitsche proved that the only cmc surface homeomorphic to a disk that meets Σ at a contact angle is either a planar disk or a spherical cap ([18]; also [20]). By the physical interest, we also study the case that the mean curvature depends linearly on a spatial coordinate.

2. Surfaces with prescribed boundary in a sphere

Let Γ be a boundary curve, possibly disconnected, on a sphere of radius ρ and centered at the origin, which will be denoted by $\mathbb{S}_\rho$. We want to study the shape of a compact embedded cmc surface S such that its boundary curve ∂S is Γ. The simplest example is when Γ is a circle $\Gamma \subset \mathbb{S}_\rho$. Then there is a planar disk ($H = 0$) spanning Γ and a family of spherical caps bounded by Γ. A second example appears when Γ lies in an open hemisphere of $\mathbb{S}_\rho$. Then, under conditions on mean convexity of Γ, it is possible to construct radial graphs on a domain of the hemisphere and spanning Γ [15,22]. By a radial graph on a domain $\Omega \subset \mathbb{S}_\rho$, also called a surface with a one-to-one central projection on Ω, we mean a surface S such that any ray emanating from the origin and crossing Ω intersects S once exactly.

We shall use the Hopf maximum principle of elliptic equations of divergence type, that in our context of cmc surfaces, we call the tangency principle.

Proposition 1 (*Tangency principle*). *Let S_1 and S_2 be two surfaces of $\mathbb{R}^3$ that are tangent at some common point p. Assume that $p \in int(S_1) \cap int(S_2)$ or $p \in \partial S_1 \cap \partial S_2$. In the latter case, we further assume that ∂S_1 and ∂S_2 are tangent at p and both are local graphs over a common neighborhood in the tangent plane $T_pS_1 = T_pS_2$. Consider on S_1 and S_2 the unit normal vectors agreeing at p. Assume that with respect to the reference system determined by the unit normal vector at p, S_1 lies above S_2 around p, which be denoted by $S_1 \geqslant S_2$. If $H_1 \leqslant H_2$ at p, then S_1 and S_2 coincide in an open set around p.*

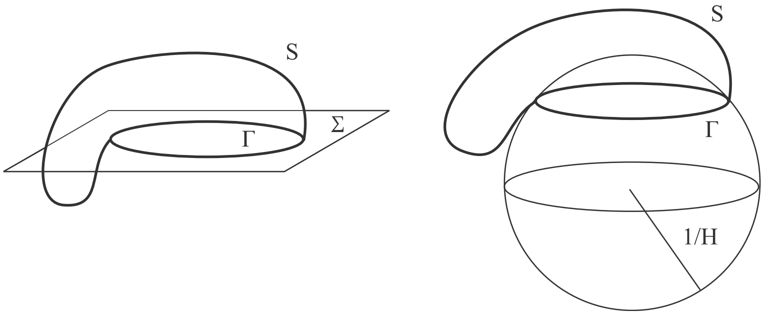

Fig. 1. Left. A possible compact embedded cmc surface S spanning a circle Γ. Right. This surface is not possible by Corollary 4.