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A new approach to generalized Fibonacci and Lucas numbers with binomial coefficients

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ABSTRACT

In this study, Fibonacci and Lucas numbers have been obtained by using generalized Fibonacci numbers. In addition, some new properties of generalized Fibonacci numbers with binomial coefficients have been investigated to write generalized Fibonacci sequences in a new direct way. Furthermore, it has been given a new formula for some Lucas numbers.

1. Introduction

For $a, b \in \mathbb{R}$ and $n > 2$, the well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$ and generalized Fibonacci $\{G_n\}$ sequences are defined by $F_n = F_{n-1} + F_{n-2},\ L_n = L_{n-1} + L_{n-2}$, and $G_n = G_{n-1} + G_{n-2}$ respectively, where $F_1 = F_2 = 1,\ L_1 = 2,\ L_2 = 1$ and $G_1 = a,\ G_2 = b$. Moreover, for the first n Fibonacci numbers, it is well known that the sum of the squares is $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ and $\sum_{i=0}^{n} \binom{n-i}{i} = F_{n+1}$. Throughout this paper, we take the notations $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Recently, there has been a huge interest of the application for Fibonacci and Lucas numbers in almost all sciences. For rich applications of these numbers in science and nature, one can see the citations in [1–8]. For instance, the ratio of two consecutive of these numbers converges to the Golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$. Applications of Golden ratio appears in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art. Benjamin et al., in [10], extended the combinatorial approach to understand relationships among generalized Fibonacci numbers. In [11], Vajda gave identities involving generalized Fibonacci numbers and binomial coefficients. All of these are special cases of the following two identities $G_{n+p} = \sum_{i=0}^{p} \binom{p}{i} G_{n-i}$ and $G_{m+(t+1)p} = \sum_{i=0}^{p} \binom{p}{i} f_t^i f_{t-1}^{p-i} G_{m+i}$. In [9], new properties of Fibonacci numbers are given and some new properties of Fibonacci numbers are investigated with binomial coefficiations. Moreover, Taskara et al., in [12], obtained new properties of Lucas numbers with binomial coefficients and gave some important consequences of these results related to the Fibonacci numbers. In [13], the authors gave a new family of k-Fibonacci numbers and established some properties of the relation to the ordinary Fibonacci numbers.

2. Main results

Fibonacci numbers arise in the solution of many combinatorial problems. In this section, we gave new formulas for Fibonacci and Lucas numbers related to generalized Fibonacci numbers and obtained some new properties of generalized Fibonacci numbers with binomial coefficients. Finally, a new formula has been given for special Lucas numbers.

The following Lemma gives new formulas for Fibonacci and Lucas numbers by using generalized Fibonacci numbers. These formulas allow us to obtain in easy form a family of Fibonacci and Lucas sequences in a new and direct way.

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Lemma 1. *For* $a^2+ab-b^2 \neq 0,\ n \in \mathbb{N}$, *we have the relations:*

(i) $F_n = \frac{aG_{n+2}-bG_{n+1}}{a^2+ab-b^2}$,

(ii) $L_n = \frac{(2a+b)G_{n+2}-(a+3b)G_{n+1}}{a^2+ab-b^2}$.

Proof.

(i) Let us use the principle of mathematical induction on n.
For $n=1$, it is easy to see that

$$F_1 = \frac{aG_3 - bG_2}{a^2+ab-b^2} = 1.$$

Assume that it is true for all positive integers $n=k$. That is,

$$F_k = \frac{aG_{k+2} - bG_{k+1}}{a^2+ab-b^2}. \tag{1}$$

Therefore, we have to show that it is true for $n=k+1$. Adding F_{k-1} to both sides of (1), we have

$$\begin{aligned} F_k + F_{k-1} &= \frac{aG_{k+2}-bG_{k+1}}{a^2+ab-b^2} + F_{k-1} = \frac{aG_{k+2}-bG_{k+1}}{a^2+ab-b^2} + \frac{aG_{k+1}-bG_k}{a^2+ab-b^2} = \frac{1}{a^2+ab-b^2}[a(G_{k+2}+G_{k+1}) - b(G_{k+1}+G_k)] \\ &= \frac{1}{a^2+ab-b^2}(aG_{k+3} - bG_{k+2}) = F_{k+1} \end{aligned}$$

as required.

(ii) The proof can be also seen by using the principle of induction on n as in (i). □

For $a=b=1$, it is obvious $F_n = G_n$. Also, for $a=2,\ b=1$, we can clearly see that $L_n = G_n$.

In the following theorem, for some values of $n \in \mathbb{N}$, we will formulate generalized Fibonacci numbers in terms of their different indices.

Theorem 2. *For* $n \in 2\mathbb{N},\ 3a-2b \neq 0$, *we have the relation*

$$G_{3n-4} = \frac{a^2+ab-b^2}{3a-2b}\sum_{i=0}^{\frac{n-2}{2}} 2^{2n-3-4i}\binom{n-2-i}{n-2-2i} - \frac{2a-b}{3a-2b}G_{3n-5}.$$

Proof. Let $c = \frac{a^2+ab-b^2}{3a-2b}$ be a real number and let us consider the sequence $\{S_k\}$. From [9], elements of this sequence can be written as

$$\begin{aligned} S_0 &= c\left[\binom{0}{0}2^1\right] = cF_3, \\ S_1 &= c\left[2^5\binom{2}{2} + 2^1\binom{1}{0}\right] = cF_9, \\ &\ \ \vdots \end{aligned} \tag{2}$$

To obtain the elements of sequence $\{S_k\}$, we can use the following method:

By forming a $(k+1)\times(k+1)$ square matrix with the rows of $\{S_k\}$ sequence and with the columns of the coefficients $c2^1,\ c2^5,\ c2^9,\ldots,c2^{2n-3}$, we see that the elements in the main diagonal and those elements above the main diagonal come from the binomial expansion as follows:

$$\begin{array}{c} \\ c2^1 \\ c2^5 \\ c2^9 \\ c2^{13} \\ c2^{17} \\ c2^{21} \\ \vdots \\ c2^{2n-3} \end{array}
\begin{array}{c} \begin{array}{cccccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & \cdots & S_k \end{array} \\
\left[\begin{array}{cccccccc}
\binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} & \cdots & \binom{\frac{n-2}{2}}{0} \\
0 & \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \cdots & \binom{\frac{n}{2}}{2} \\
0 & 0 & \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \cdots & \binom{\frac{n}{2}+1}{4} \\
0 & 0 & 0 & \binom{6}{6} & \binom{7}{6} & \binom{8}{6} & \cdots & \binom{\frac{n}{2}+2}{6} \\
0 & 0 & 0 & 0 & \binom{8}{8} & \binom{9}{8} & \cdots & \binom{\frac{n}{2}+3}{8} \\
0 & 0 & 0 & 0 & 0 & \binom{10}{10} & \cdots & \binom{\frac{n}{2}+4}{10} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \binom{n-2}{n-2}
\end{array}\right] \end{array}$$