Applied Mathematics and Computation 220 (2013) 518–535

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Derivation of identities involving some special polynomials and numbers via generating functions with applications

Moawwad El-Mikkawy *, Faiz Atlan

Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

ARTICLE INFO

Keywords:
Bernoulli numbers
Bernoulli polynomials
Euler numbers
Euler polynomials
Genocchi numbers
Genocchi polynomials
Generating functions
Maple

ABSTRACT

The current article focus on the ordinary Bernoulli, Euler and Genocchi numbers and polynomials. It introduces a new approach to obtain identities involving these special polynomials and numbers via generating functions. As an application of the new approach, an easy proof for the main result in [6] is given. Relationships between the Genocchi and the Bernoulli polynomials and numbers are obtained. Some interesting identities are discovered.

1. Introduction and basic definitions

The use of polynomials in many areas of science and engineering is quite remarkable: numerical analysis, operator theory, special functions, complex analysis, statistics, sorting and data compression, etc. Throughout this paper we shall focus on three special polynomials. These useful polynomials are the Bernoulli, Euler and Genocchi polynomials. The Bernoulli polynomials and the Bernoulli numbers, for example, are of fundamental importance in several parts of analysis and in the calculus of finite difference. These polynomials and numbers have applications in many fields. For example, in numerical analysis, statistics and combinatorics. The interested reader may refer to [1,2,4,6–9,11,15–17,19,29,31,41,44] and the references therein. The basic properties of Bernoulli numbers and polynomials are well known and are outlined, for example in [5,10,12,14,22,34,39,42].

The generating functions have an important role in many branches of mathematics, statistics and computer science, see for example [26,27,36,43].

The main object of the current paper is to show that the generating functions approach can be employed efficiently to obtain old and new identities involving the Bernoulli, Euler and Genocchi polynomials and numbers.

The current paper is organized as follows: In the next section, we list some important properties, without proofs, for some exponential generating functions. The main results of this paper are given in Sections 3 and 4. Finally, a conclusion is given in Section 5.

Throughout the paper, δ_{nm} is the kronecker symbol which is equal to 1 or 0 according as $n = m$ or not. Also empty summation is assumed equal to zero.

Definition 1.1 [36]. The falling factorial of x, $(x)_n$ is defined by

* Corresponding author.
E-mail addresses: m_elmikkawy@yahoo.com, mikkawy@mans.edu.eg (M. El-Mikkawy), faizatlan11@yahoo.com (F. Atlan).

0096-3003
http://dx.doi.org/10.1016/j.amc.2013.06.014

$$(x)_n = \begin{cases} 1 & \text{if} \quad n = 0, \\ x(x-1)(x-2)\ldots(x-n+1) & \text{if} \quad n \geqslant 1. \end{cases}$$

Definition 1.2 [13]. Let f is a real valued function. The forward difference operator, Δ of f is defined by

$$\Delta f(x) = f(x+1) - f(x).$$

Generating functions are one of the most surprising, useful, and clever tools in mathematics, computer science and statistics. By using generating functions, we can transform problems about sequences which they generate into problems about real valued functions.

Definition 1.3 [36]. The ordinary generating function (OGF) of a sequence $< a_0, a_1, a_2, \ldots >$ is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Definition 1.4 [26]. The exponential generating function (EGF) of a sequence $< a_0, a_1, a_2, \ldots >$ is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Throughout this paper, we shall be concerned with the exponential generating functions. For these generating functions, we have

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (a_n + b_n) \frac{x^n}{n!} \quad \text{and} \tag{1}$$

$$\left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}, \quad \text{where} \quad c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \quad n \geqslant 0. \tag{2}$$

If $g(x)$ is the generating function of the sequence $< a_0, a_1, a_2, \ldots >$, then we may write the correspondence between the sequence and its generating function by using a double-sided arrow as follows

$$< a_0, a_1, a_2, \ldots > \leftrightarrow g(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{3}$$

Definition 1.5 [18]. The sequences $< B_0, B_1, B_2, \ldots >$ of the Bernoulli numbers, B_n, $n \geqslant 0$ and $< B_0(x), B_1(x), B_2(x), \ldots >$ of the Bernoulli polynomials, $B_n(x)$, $n \geqslant 0$ are defined by the exponential generating functions

$$G(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{4}$$

and

$$F(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{5}$$

respectively.

The first 6 Bernoulli polynomials are

$$B_0(x) = 1 \quad B_1(x) = x - \frac{1}{2} \quad B_2(x) = x^2 - x + \frac{1}{6} \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$$

The first 11 Bernoulli numbers are

$$B_0 = 1 \quad B_1 = -\frac{1}{2} \quad B_2 = \frac{1}{6} \quad B_3 = 0 \quad B_4 = -\frac{1}{30} \quad B_5 = 0$$

$$B_6 = \frac{1}{42} \quad B_7 = 0 \quad B_8 = -\frac{1}{30} \quad B_9 = 0 \quad B_{10} = \frac{5}{66}.$$