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Symmetric positive solutions to singular system with multi-point coupled boundary conditions

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ABSTRACT

In this paper, we study the existence and multiplicity of symmetric positive solutions for a nonlinear system with multi-point coupled boundary conditions. The arguments are based upon a specially constructed cone and the fixed point index theorem in cones. An example is then given to demonstrate the applicability of our results.

1. Introduction

Coupled boundary conditions (BCs) arise in the study of reaction–diffusion equations, Sturm–Liouville problems, mathematical biology and so on, see [1–17] and [18, Chapter 13]. In [4], Cardanobile and Mugnolo studied the following parabolic system

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x), \quad t \geqslant 0,\ x \in \Omega \in \mathbb{R}^n$$

with a general class of coupled BCs of the form

$$f_{|\partial\Omega} \in \mathcal{Y}, \quad \frac{\partial f}{\partial v} \in \mathcal{Y}^{\perp}$$

where $\mathcal{Y}$ is a closed subspace of $L^2(\partial\Omega; W)$, the unknown function u takes values in a separable Hilbert space W and $\mathcal{L}$ is an elliptic operator with operator-valued symbol.

Delgado et al. [5] investigated the following system with coupled BCs of the type

$$\begin{aligned}
&u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \mu u(1-u) \quad \text{in } \Omega\times(0,T),\\
&0 = \Delta v - v + \frac{u}{1+u} \quad \text{in } \Omega\times(0,T),\\
&\frac{\partial u}{\partial n} - \chi u\frac{\partial v}{\partial n} = r(\theta - u) \quad \frac{\partial v}{\partial n} = r'\left(\frac{\theta}{2} - v\right) \quad \text{in } \Omega\times(0,T),\\
&u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}$$

where μ, r, r', θ, χ denote nonnegative constants. The authors proved the existence of a unique positive global in time classical solution and analyzed the associated stationary problem.

Leung [7] studied the following reaction–diffusion system for prey-predator interaction

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$$\begin{cases} u_t(t,x) = \sigma_1 \Delta u + u(a + f(u,v)), & t \geqslant 0,\ x \in \Omega \subset \mathbb{R}^n, \\ v_t(t,x) = \sigma_2 \Delta v + v(-r + g(u,v)), & t \geqslant 0,\ x \in \Omega \subset \mathbb{R}^n, \end{cases}$$

subject to the coupled BCs

$$\frac{\partial u}{\partial \eta} = 0, \quad \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \quad \text{on } \partial\Omega,$$

where the functions $u(t,x), v(t,x)$ respectively represent the density of prey and predator at time $t \geqslant 0$ and at position $x = (x_1, \ldots, x_n)$. Similar coupled BCs are also studied in [2] for biochemical system.

At the same time, the theory of multi-point boundary value problems (BVPs) for ordinary differential equations arises in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVPs; many problems in the theory of elastic stability can be handled as multi-point BVPs too. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations have received a great deal of attention. To identify a few, we refer the readers to [19,18–26] and the reference therein for three-point BVPs and [27–31] for multi-point BVPs.

Very recently, Asif and Khan [32] studied the following a coupled singular system subject to four-point coupled BCs of the type

$$\begin{cases} -x''(t) = f(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = g(t, x(t), y(t)), & t \in (0,1), \\ x(0) = 0, \quad x(1) = \alpha y(\xi), \\ y(0) = 0, \quad y(1) = \beta x(\eta), \end{cases} \tag{1.1}$$

where the parameters α, β, ξ, η satisfy $\xi, \eta \in (0,1), 0 < \alpha\beta\xi\eta < 1, f, g : (0,1) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and singular at $t = 0, t = 1$. The authors obtained at least one positive solution to the system (1.1) by using Guo–Krasnosel'skii fixed-point theorem.

We notice that a type of symmetric problem has received much attention, for instance, [19,27–29,31–34,**?**,36], and the references therein. In [33], Sun discussed the existence and multiplicity of the BVPs

$$u''(t) + h(t)f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.2}$$

$$u(t) = u(1-t), \quad u'(0) = u'(1) = u(1/2), \tag{1.3}$$

by applying the Krasnosel'skii fixed-point theorem. Sun [34] also considered the optimal existence criteria for symmetric positive solutions to the Eq. (1.2) with the following BCs: $u(0) = u(1) = \alpha u(\eta)$. Recently, Wang and Sun [35] studied the existence of symmetric positive solutions to Eq. (1.2) with the following BCs: $u(t) = u(1-t), \alpha u'(0) + \beta u'(1) = \gamma u(1/2)$. Inspired by the above mentioned work and wide applications of coupled BCs in various fields of sciences and engineering, we study the existence of symmetric positive solutions to a singular system

$$\begin{cases} -x''(t) = a_1(t) f(t, x(t), y(t)), & t \in (0,1), \\ -y''(t) = a_2(t) g(t, x(t), y(t)), & t \in (0,1), \\ x(0) = \sum_{i=1}^{m} \alpha_i y(\xi_i), \quad x(1) = \sum_{i=1}^{m} \alpha_i y(\bar{\xi}_i), \\ y(0) = \sum_{i=1}^{m} \beta_i x(\eta_i), \quad y(1) = \sum_{i=1}^{m} \beta_i x(\bar{\eta}_i), \end{cases} \tag{1.4}$$

where $m \geqslant 1$ is an integer, the parameters $\alpha_i, \beta_i > 0,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_m < \frac{1}{2}$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_m < \frac{1}{2}, \xi_i + \bar{\xi}_i = 1, \eta_i + \bar{\eta}_i = 1\ (i = 1, 2, \ldots, m)$. We assume $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and $f(\cdot, x, y), g(\cdot, x, y)$ are symmetric on $[0,1]$ for all $x, y \in [0,\infty), a_1, a_2 \in C(0,1)$ are symmetric on $(0,1)$, and may be singular at $t = 0$ and $t = 1$. By a symmetric positive solution of the system (1.4), we mean that $(x,y) \in C[0,1] \cap C^2(0,1) \times C[0,1] \cap C^2(0,1), (x,y)$ satisfies (1.4), x, y are symmetric and $x(t) > 0, y(t) > 0$ for all $t \in [0,1]$. To the best knowledge of the author, there is no earlier literature studying this problem. This paper attempts to fill part of this gap in the literatures.

The rest of the paper is organized as follows. In Section 2, we present a positive cone, a fixed point theorem which will be used to prove existence of symmetric positive solutions, Green's function for the system of BVPs (1.4) and some related lemmas. In Section 3, we present main results of the paper and finally an example to illustrate the application of our main results.

2. Preliminaries and lemmas

The basic space used in this paper is $E = C[0,1] \times C[0,1]$. Obviously, the space E is a Banach space if it is endowed with the norm as follows: