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Asymptotic convergence of cubic Hermite collocation method for parabolic partial differential equation

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ABSTRACT

In this paper, the asymptotic convergence of cubic Hermite collocation method in continuous time for the parabolic partial differential equation is established of order $O\left(h^2\right)$. The linear combination of cubic Hermite basis taken as approximating function is evaluated using the zeros of Chebyshev polynomials as collocation points. The theoretical results are verified for two test problems.

1. Introduction

Over the years, mathematicians and engineers are engaged in analyzing the second order parabolic partial differential equations (PDEs) of the form:

$$\frac{\partial C}{\partial T} = m_1 \frac{\partial^2 C}{\partial Z^2} - m_2 \frac{\partial C}{\partial Z}, Z \in (0,1), \tag{1}$$

for the Dirichlet, Neumann or Robin boundary conditions:

$$m_3 C + m_4 \frac{\partial C}{\partial Z} = \phi_1, Z = 0, T > 0, \tag{2}$$

$$m_5 C + m_6 \frac{\partial C}{\partial Z} = \phi_2, \quad Z = 1, \quad T > 0, \tag{3}$$

and initial condition:

$$C(Z,0) = \phi_3. \tag{4}$$

Here $m's$ are real numbers lying between 0 and 1 and $\phi's$ are non negative real numbers. Z is bounded in open interval $(0,1)$ on real axes. The operator H defined by $\frac{\partial^2}{\partial Z^2}$ in spatial and time domains, is positive definite in $L_2(0,1)$, space of all real valued Lebesgue measurable functions square integrable $(0,1)$, for all $T > 0$. Hence, there exist a continuous function $w(Z,T) \in D(H)$, space of all the functions twice continuously differentiable function with compact support on the interval [0,1] as reported in literature [1], such that for every constant λ:

$$\lambda\langle w,w\rangle \leqslant \langle Hw,w\rangle, \quad \text{for } Z \in (0,1) \text{ and } T > 0, \tag{5}$$

and

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$$\langle Hw, w\rangle = \langle w, Hw\rangle, \quad \forall\, Z, T. \tag{6}$$

The inner product in $L_2(0,1)$ is defined as $\langle p,q\rangle = \int_0^1 p(x)q(x)dx.$ (7)

The Lebesgue norm is defined as $\|C\| = \left(\int_0^1 C^2\right)^{1/2}.$ (8)

The Eucledian norm is defined as $\|C\| = \left(\sum_{i=1}^{N} C_i^2\right)^{1/2}.$ (9)

The definition given in [2] considers a family of mathematical problems parametrized by singular perturbation parameter υ, where υ lies in the semi open interval $0 < \upsilon \leqslant 1$. It is assumed that each problem in the family has the unique solution denoted by u_υ and each u_υ is approximated by a sequence of numerical solutions $\{(U_\upsilon, \bar{\Omega}^M)\}_{M=1}^{\infty}$ where U_υ is defined on the $\bar{\Omega}^M$ representing the set of points in R and M is the discretization parameter. Therefore, the numerical solutions U_υ are said to converge to the exact solution u_υ, if their exist a positive integer M_0 and positive numbers K and p (where M_0, K and p are all independent of M and υ) such that for all $M \geqslant M_0$,

$$\sup_{0<\upsilon\leqslant 1} \|U_\upsilon - u_\upsilon\|_{\bar{\Omega}^M} \leqslant KM^{-p},$$

where p is the rate of convergence and K is the error constant.

By making suitable modifications the system (1-4) has been extended to mass transfer between the solid and liquid phases [3], separation of glycerol from biodiesel [4], tubular flow reactor [5–7], analysis of distillation column [8], design and development of airlift loop reactors [9], heat transfer processes in gas–solid turbulent fluidized systems [10], chromatography [11], measurement of neutron flux [12], sorption [13], enzymatic hydrolysis of racemic ibuprofen ester by lipase [14], desorption [15], modeling and design of optoelectronic devices [16], one-dimensional heat equation [17–21], nonlinear dynamic [22] and nuclear scattering problems [23].

No reports are found in the literature regarding the asymptotic convergence of cubic Hermite collocation method (CHCM). Therefore, an attempt is made to present the convergence analysis of CHCM for parabolic partial differential equations.

2. Cubic Hermite collocation method

Related to the partition $a = x_0 < x_1 < \ldots < x_n = b$, the cubic Hermite interpolant of function f is a function s, that satisfies [24]:

(1) on each subinterval $[x_j, x_{j+1}]$, s coincides with a cubic polynomial $s_j(x)$,
(2) s interpolates f and f' at $x_0, x_1, \ldots, x_n$,
(3) s and s' are continues on [a, b].

The Hermite cubic Interpolation of f and its first derivative at $x = x_j$ requires:

$$s_j(x_j) = f(x_j) \text{ and } s_j'(x_j) = f'(x_j).$$

Combining the continuity of s and s' at $x = x_{j+1}$ with interpolation of f and its first derivative at $x = x_{j+1}$, one gets:

$$s_j(x_{j+1}) = s_{j+1}(x_{j+1}) = f(x_{j+1}) \quad \text{and} \quad s_j'(x_{j+1}) = s_{j+1}'(x_{j+1}) = f'(x_{j+1}).$$

Therefore, $s_j(x)$ is a third degree polynomial that interpolates both f and f' at $x = x_j$ and at $x = x_{j+1}$. Hence, an arbitrary cubic polynomial defined on [0,1] may be written as:

$$s_j(x) = H_1(x)f(x_j) + H_2(x)f'(x_j) + H_3(x)f'(x_{j+1}) + H_4(x)f(x_{j+1}), \tag{10}$$

which is associated with two Hermite cubic polynomials given below:

$$H_{2j-1}(x) = \begin{cases} \left(\frac{x-x_{j-1}}{h_{j-1}}\right)^2\left(3 - \frac{2(x-x_{j-1})}{h_{j-1}}\right); & x \in [x_{j-1}, x_j] \\ \left(1 - \frac{x-x_j}{h_j}\right)^2\left(1 - \frac{2(x-x_j)}{h_j}\right); & x \in [x_j, x_{j+1}] \\ 0; & otherwise \end{cases},$$

$$\text{and } H_{2j}(x) = \begin{cases} -h_{j-1}\left(\frac{x-x_{j-1}}{h_{j-1}}\right)^2\left(1 - \frac{x-x_{j-1}}{h_{j-1}}\right); & x \in [x_{j-1}, x_j] \\ h_j\left(1 - \frac{x-x_j}{h_j}\right)^2\left(\frac{x-x_j}{h_j}\right); & x \in [x_j, x_{j+1}] \\ 0; & otherwise \end{cases}.$$