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An effective method for solving nonlinear equations and its application

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ABSTRACT

The linearized partial differential equation from the nonlinear partial differential equation which was proposed by Rudin, Osher and Fatemi [L. I. Rudin, S. Osher and E. Fatemi, *Nonlinear total variation based noise removal algorithms*] for solving image decomposition was introduced by Chambolle [A. Chambolle, *An algorithm for total variation minimization and applications*] and R. Acar and C.R. Vogel [R. Acar and C. R. Vogel, *Analysis of bounded variation penalty methods for ill-posed problems*]. In this paper, we propose a method for solving the linearized partial differential equation and we show numerical results for denoising which demonstrate a significant improvement over other previous works.

1. Introduction

Assume that $u_0 = u + v$, where u_0 is a given image, u is the true signal and v is the noise or texture image, and all these variables are defined on an open and bounded domain $\Omega \subset R^2$. The goal of image denoising is to find the true solution u, and u is obtained by solving a partial differential equation and satisfying the minimizer of the functional which is constructed for obtaining the goal. Many methods for image denoising have been studied [33,31,35,34,3,36,14,15,9,10,1,19,26,24,4,5,8,12,7].

$TV - L^2$ (**ROF**) is the well known edge preserving model proposed by Rudin, Osher and Fatemi [31]. This model decomposes an image u_0 into a component u belonging to the space of functions of bounded variation notated by BV and a component v in L^2, where BV space is introduced by R. Acar and C.R. Vogel [1]. The ROF model is

$$\inf_u \{T(u)\} = \inf_u \left\{ \lambda \int_\Omega |\nabla u| + \frac{1}{2}\|u - u_0\|^2 \right\},$$

where λ is a weight parameter, and $\|u - u_0\|$ is a fidelity term. If $u \in L^1(\Omega)$ and $\int_\Omega |\nabla u| < \infty$, then $u \in BV(\Omega)$. Assume that u is the minimizer of ROF's functional, $T(u)$. By using the Euler–Lagrange equation, the first derivation of T is

$$g_0(u) = -\alpha\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + u - u_0. \tag{1.1}$$

However, this is a nonlinear equation which has a highly nonlinear and non-differentiable term. It is not easy to solve such nonlinear equations [21,17,16,27,28].

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A useful method to solve nonlinear equations is Chambolle's method. In Chambolle's work, the total variation, which is the first term of $T(u)$, is changed into a dual form by using the Legendre-Fenchel transform as follows:

$$\inf_{\xi\in K}\left\{\int_\Omega (\alpha\mathrm{div}\xi - u_0)^2 dx\right\}, \tag{1.2}$$

where the set of test functions is

$$K = \{\mathrm{div}\xi : \xi \in C_c^1(\Omega; R^2), \quad |\xi(x)| \leqslant 1, \quad \forall x \in \Omega\}.$$

The problem (1.2) can be solved by the optimality condition and the *semi-implicit gradient descent (fixed point) iteration scheme*

$$\frac{\xi^{n+1} - \xi^n}{\triangle t} = \nabla(\alpha\mathrm{div}\xi^n - u_0) - |\nabla(\alpha\mathrm{div}\xi^n - u_0)|\xi^{n+1} \Rightarrow \xi^{n+1} = \frac{\xi^n + \triangle t\nabla(\alpha\mathrm{div}\xi^n - u_0)}{1 + \triangle t|\nabla(\alpha\mathrm{div}\xi^n - u_0)|}.$$

Thus, the numerical solution of the ROF model is obtained from the condition $u = u_0 - \alpha\xi$. The duality is a useful method for linearizing the nonlinear differential term in $T(u)$. However, Chambolle's method has to use division and division is concomitant with roundoff error, $1/3 \neq 0.333333$. This prevents an exact numerical solution.

To overcome these problems, we propose a method for solving a linearized partial differential equation without using an inverse matrix in image decompositions. Our method can be applied to solve several image decomposition models, for instance, the $TV - H^{-1}$ model (the OSV model).

In Section 2, we describe our idea, which involves using a sequence and analysis convergence rate, and show the limit of our sequence satisfies the minimizer of the models in which we are interested. In Section 3, we demonstrate our results and logic through numerical experiments.

2. Method

In this section, we describe our algorithm to decompose the true solution and the noise from a given image $u_0 \in L^2(\Omega)$. Using a dual form of the total variation, we obtain a linear operator and construct a functional from this linear operator, which is a quadric form. The limit of our constructed sequence minimizes the constructed functional which is equivalent to the functional of the models of image decomposition. Particularly, we have the asymptotic rate of the convergence of the sequence which depends on a search direction, z_n, and a parameter, α_n, called the step-size as the conjugate gradient method or the Richardson's method [25,2,37]. The constructed search directions in our algorithm reduce errors. Thus, our sequence successfully reduces error and its limit is the minimizer of the constructed functional in image decomposition.

2.1. Duality

We apply the duality to the ROF model or the OSV model, in order to obtain a linear operator, and take the set of test functions

$$K = \{\mathrm{div}\xi : \xi \in C_c^1(\Omega; R^2), |\xi(x)| \leqslant 1 \forall x \in \Omega\}.$$

The ROF model can be set in the dual formulation

$$\inf_u \sup_{\xi\in K}\left\{-\int_\Omega u(x)\mathrm{div}\xi(x)dx + \frac{1}{2\lambda}\int_\Omega (u - u_0)^2\right\}. \tag{2.1}$$

Here, for each ξ, a minimizer u of (2.1) has the form $u = u_0 + \lambda\mathrm{div}\xi$. Substituting this form for u back into (2.1) reformulates the problem into

$$\begin{aligned} &\sup_{\xi\in K} \quad \left\{\int_\Omega \left(-\tfrac{\lambda}{2}(\mathrm{div}\xi)^2 - u_0\mathrm{div}\xi\right)\right\} \\ &-\inf_{\xi\in K} \quad \left\{\int_\Omega \left(\tfrac{\lambda}{2}(\mathrm{div}\xi)^2 + u_0\mathrm{div}\xi\right)\right\} \end{aligned} \tag{2.2}$$

Finding $\xi \in K$ satisfying (2.2) is the same problem with finding ξ satisfying (2.3) because the last term of (2.3) is constant.

$$-\inf_{\xi\in K}\left\{\int_\Omega \left(\frac{\lambda}{2}(\mathrm{div}\xi)^2 + u_0\mathrm{div}\xi\right) + \frac{u_0^2}{2\lambda}\right\} \tag{2.3}$$

Thus, the constrained optimization problem (2.2) is equivalent to the following:

$$\inf_{\xi\in K}\left\{\int_\Omega \left(\mathrm{div}\xi + \frac{u_0}{\lambda}\right)^2 dx\right\}. \tag{2.4}$$

Chambolle shows that the functional (2.4) has a unique minimizer using a convergent sequence constructed by the semi-implicit gradient descent (fixed point) iteration scheme. From the existence of the minimizer, we can obtain a problem as