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Laplace Transform and finite difference methods for the Black–Scholes equation

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ABSTRACT

In this paper we explore discrete monitored barrier options in the Black–Scholes framework. The discontinuity arising at each monitoring data requires a careful numerical method to avoid spurious oscillations when low volatility is assumed. Here a technique mixing the Laplace Transform and the finite difference method to solve Black–Scholes PDE is considered. Equivalence between the Post–Widder inversion formula joint with finite difference and the standard finite difference technique is proved. The mixed method is positivity-preserving, satisfies the discrete maximum principle according to financial meaning of the involved PDE and converges to the underlying solution. In presence of low volatility, equivalence between methods allows some physical interpretations.

1. Introduction

One of the main concerns about financial options is what the exact values of the options are. In absence of evaluation formula for non-standard options, numerical technique is required. Usually the choice goes toward numerical methods with high order of accuracy (for instance in the finite difference method the Crank–Nicolson scheme) and no attention is paid to the fact how the financial provision of the contract can affect the reliability of the numerical solution. Special options, as discretely monitored barrier options are characterized by discontinuities that are renewed at each monitoring date. In presence of *low volatility* the Black–Scholes PDE becomes *convection dominated*. As a consequence, numerical diffusion or spurious oscillations may arise, so that special numerical techniques have to be employed. A viable route to circumvent discontinuity issues is considering an Integral Transforms method. If we assume that the volatility σ of the underlying asset price S and the risk-free interest rate r of the market depends only on S, i.e., $\sigma = \sigma(S), r = r(S)$, then the Laplace Transform becomes a useful tool. The Black–Scholes equation is solved by the Laplace Transform method for time t discretization. The resulting ordinary differential equation (ODE) is solved by a finite difference scheme and, as a final result, a Laplace Transform of the solution is obtained. Hereinafter we call that method a *'mixed method'*. The crucial issue is the Laplace Transform inversion. A lot of methods are available in literature [1]. They can be roughly classified into two categories: the ones using *complex values* of the Laplace Transform and the ones using only *uniquely real values*. Here, rather than proposing a new method for the Laplace Transform inversion on the real axis, we consider the well-known Post–Widder inversion formula [1, p. 37 and 141–3]. Next we prove the equivalence between *standard finite difference schemes* and the *mixed method* of Laplace Transform with the Post–Widder inversion formula jointly with special finite difference schemes that solve the resulting ODE. We prove that *the mixed method is positivity-preserving, satisfies the discrete maximum principle, is spurious oscillations free, convergent to exact solution and finally provides a physical meaning to Post–Widder inversion formula.*

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In order to make our analysis concrete, we concentrate our attention on a double barrier knock-out call option with a discrete monitoring clause. Such option has a payoff condition equal to $\max(S-K,0)$, where K denotes the strike price, but the option expires worthless if before the maturity T the asset price has fallen outside the corridor $[L,U]$ at the prefixed monitoring dates $0=t_0<t_1<\cdots<t_F=T$. In the intermediate periods the Black Scholes equation is applied over the real positive domain. The discontinuity in the initial conditions will be renewed at every monitoring date and often the Crank–Nicolson numerical solution is affected by spurious oscillations that do not decay quickly [2–4]. The presented analysis can be easily extended to many other exotic contracts (digital, supershare, binary and truncated payoff options, callable bonds and so on).

2. Some background

2.1. The Black–Scholes PDE

Let $V(S,t)$ be the value of an option, where S is the current value of the underlying asset and t is the time to expiry T. The value of the option is related to the current value of the underlying asset via two stochastic parameters, the volatility $\sigma=\sigma(S)$ and the interest rate $r=r(S)$ of the Black–Scholes equation. The price $V(S,t)$ of the option satisfies the Black–Scholes partial differential equation [5]

$$-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{2.1}$$

endowed with initial and boundary conditions:

$$V(S,0)=\max(S-K,0)\,1_{[L,U]}(S) \tag{2.2}$$

$$V(S,t)\to 0 \text{ as } S\to 0 \text{ or } S\to\infty, \tag{2.3}$$

with updating of the initial condition at the monitoring dates $t_i, i=1,\cdots,F$:

$$V(S,t_i)=V(S,t_i^-)1_{[L,U]}(S),\quad 0=t_0<t_1<\cdots<t_F=T, \tag{2.4}$$

where $\mathbf{1}_{[L,U]}(S)$ is the indicator function, i.e.,

$$\mathbf{1}_{[L,U]}=\begin{cases}1 & \text{if}\quad S\in[L,U]\\ 0 & \text{if}\quad S\notin[L,U].\end{cases}$$

Here K represents the strike price. The knock-out clause at the monitoring date introduces a *discontinuity* at the barriers set at $S=L$ and $S=U$ respectively. The presence of undesired spurious oscillations is frequently observed near the barriers and near the strike, if unsuitable finite difference schemes are used. These spikes, which remain well localized, do not reflect instability but rather that the discontinuities that are periodically produced by the barriers at monitoring dates. The spikes cannot decay fast enough.

The parabolic nature of the Black–Scholes equation ensures that, being the initial condition $V(S,0)=(S-K)^+1_{[L,U]}(S)$ square–integrable, the solution is smooth in the sense that $V(\cdot,t)\in C^\infty(\mathbb{R}^+),\forall t\in(t_{i-1},t_i^-],\ i=1,\ldots,F$. Thus rough initial data give rise to smooth solutions in infinitesimal time.

The smoothness of $V(S,t), t>0$, allows us to invert the Laplace Transform via Post–Widder inversion formula for calculating $V(S,T)$.

As a consequence of the parabolic nature of Black–Scholes equation, the solution obeys the maximum principle:

$$\max_{S\in[0,+\infty]}|V(S,t_1)|\geqslant\max_{S\in[0,+\infty]}|V(S,t_2)|,\quad t_1\leqslant t_2. \tag{2.5}$$

This inequality means that the maximum value of $V(S,t)$ should not increase as t increases.

2.2. The Laplace Transform

Here some particular definitions used in the sequel are given. If for a function $f:\mathbb{R}^+\to\mathbb{R}^+, f:t\to f(t)$ there exists a $\lambda_0\in\mathbb{R}$, such that the Laplace Transform of the function $f(t)$ defined by

$$F(\lambda)=\mathcal{L}[f](\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt, \tag{2.6}$$

exists for all $\lambda\in\mathbb{R},\ \lambda>\lambda_0$, then $F(\lambda)$ is the Laplace Transform of f.

If $f:\mathbb{R}^+\to\mathbb{R}^+, f:t\to f(t)$ is of exponential order, i.e., for some $\lambda_0\in\mathbb{R}$

$$\sup_{t>0}f(t)e^{-\lambda_0 t}<\infty, \tag{2.7}$$

then the Laplace Transform (2.6) exists for all $\lambda>\lambda_0$ and it is infinitely differentiable with respect to λ for $\lambda>\lambda_0$.