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A second-order positivity preserving numerical method for Gamma equation

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ABSTRACT

In this work we consider Cauchy problem for the so called Gamma equation, which can be derived by transforming the fully nonlinear Black–Scholes equation for option price into a quasi-linear parabolic equation for the second derivative (*Greek*) $\Gamma = V_{SS}$ of the option price V. We develop an efficient positivity preserving explicit numerical method for solving the model problem concerning different volatility terms. We prove that the obtained semi-discretization is positive and the corresponding full approximation also preserves this property, if the time step is restricted. The stability of the difference scheme in L_1 norm is shown and the existence of interface curves is investigated numerically. Results of numerical simulations are given and discussed.

1. Introduction

In recent years there has been an increasing interest in problems arising in Financial Engineering. While the standard Black–Scholes model was the single most important step in the development of modern derivative asset analysis, the underlying assumptions of constant volatility and of a perfectly liquid market are clearly at odds with reality. Often prices and hedging strategies in these models are described by fully nonlinear versions of the standard Black–Scholes parabolic equation; see, for instance [2,4,9,10,18,20,29]. It turns out that despite substantial differences in the underlying financial framework, these non-linear Black–Scholes equations have a very similar structure, making them a useful tool for measuring the risk management cost for a (book of) derivatives in illiquid markets or in markets with stochastic volatility.

A variety of numerical methods were used in previous papers for studying properties of typical non-linear Black–Scholes equations, see, for instance [2,7,8,12,13,18,21,25,29] and references there in.

In this paper we will be concerned with several models from the more relevant class of nonlinear Black–Scholes equations for European options with volatility depending on different factors, such as the stock price, the time, the option price and its derivatives, where the nonlinearity results from the presence of transaction cost.

Higher derivative of the option price also provide important information about the behavior of the option. These quantities are known in the financial literature as the *Greeks*. Due to the Greeks being relevant for the quantitative analysis, reliable numerical methods are required for the pricing of options which not only provide a good approximation for the price, but also for its derivatives.

A common practice is first to construct numerical methods for evaluation the option price and on this base to obtain estimates for the Greeks [2,5–8,12,29,30]. An efficient approach for the computation of the fair value of a basket option as well as some of its Greeks are presented in [12]. In order to estimate the Greeks, the authors construct a piecewise multi-linear interpolant of the pricing function with respect to the market parameters and compute its partial derivatives.

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We pay attention to the Gamma equation, i.e. the PDE satisfied by the function $S\partial_{SS}V(S,t)$, where the second derivative $\partial_{SS}V(S,t)$ is the so-called Γ of an option, S is the underlying asset, t is the time and $V(S,t)$ is the price of the option.

We present a positivity preserving, second-order explicit finite difference scheme for Cauchy problem of Gamma equation for the models described in Section 2. The scheme is a combination of the finite volume, the van Leer flux-limiter technique and adaptive mesh refinement in time.

The organization of the exhibition is as follows. In Section 2 we proceed with the study of the properties of continuous solutions. The numerical method is described in the next section. In Section 4 we analyze the numerical scheme, concerning positivity and stability. Existence of interface curve, when the initial data is compact supported is numerically investigated in Section 5. Finally, in Section 6 we present an discuss numerical experiments.

2. Non-linear Black–Scholes equations

In this section we shall present three typical non-linear models and the corresponding Gamma equations.

2.1. Non-linear models

The solution of famous Black–Scholes equation

$$\frac{\partial V}{\partial t_*}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,\quad 0\leqslant S<\infty,\ 0\leqslant t_*\leqslant T \tag{1}$$

provides both an option pricing formula for European option and a hedging portfolio that replicates the contingent claim under too restrictive for the practice assumptions [2,10,18,29]. On the other hand, if transaction costs, feedback effects from the trading activity, market illiquidity are not neglected, the linear Black–Scholes equation is replaced by a nonlinear one.

2.1.1. Barles–Soner model

Barles and Soner [2] derived more complicated model by following the utility function approach and prove that as ϵ and μ go to $0, V$ is the unique (viscosity) solution of (1) with volatility term, given by

$$\sigma^2=\sigma_{BS}^2=\sigma_0^2[1+\Psi(e^{r(T-t_*)}a^2S^2V_{SS})],$$

where σ_0 is the volatility of the underlying asset, $a=\mu/\sqrt{\epsilon}$ is a parameter to measure transaction cost and risk aversion, $\Psi(x)$ solves the ordinary differential equation

$$\Psi(x)=\frac{\Psi(x)+1}{2\sqrt{x\Psi(x)}-x},\quad x\neq 0,\ \Psi(0)=0.$$

The analysis of this ordinary differential equation by Barles and Soner implies that

$$\lim_{x\to\infty}\frac{\Psi(x)}{x}=1\quad\text{and}\quad\lim_{x\to-\infty}\Psi(x)=-1.$$

This property allows the treatment of the function $\Psi(\cdot)$ as the identity for large arguments and therefore the volatility becomes

$$\sigma_{BS}^2=\sigma_0^2[1+e^{r(T-t_*)}a^2S^2V_{SS}].$$

2.1.2. Risk adjusted pricing methodology

In this model, proposed by Kratka [20] and improved by Jandačka–Ševčovič in [18], the optimal time-lag δt_* between the transactions is found to minimize the sum of the rate of the transaction costs and the rate of the risk from an unprotected portfolio. That way the portfolio is still well protected and the modified volatility is now of the form

$$\sigma^2=\sigma_{JS}^2=\sigma_0^2\left(1+\mu(SV_{SS})^{\frac{1}{3}}\right),$$

where $\mu=3(C^2M/(2\pi))^{1/3},\ M\geqslant 0$ is the transaction cost measure and $C\geqslant 0$ is the risk premium measure.

2.1.3. Frey model

Assuming that the trading action of a large investor will affect the price of the underlying asset, then the modified volatility $\sigma^2=\sigma_{FP}^2$ (Frey model [10] among others) is

$$\sigma_F^2=\frac{\sigma_0^2}{1-\rho SV_{SS}}.$$

Here ρ is a parameter measuring the market liquidity.