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Characterization of efficient frontier for mean–variance model with a drawdown constraint ☆

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ABSTRACT

This paper examines an optimal portfolio selection problem with a drawndown constraint based on mean–variance model when: (1) there is no arbitrage; (2) the market is complete; (3) there are finitely many states. First, we obtain the necessary and sufficient condition for the existence of the optimal solution to the problem. Then we prove that the efficient frontier of the model is a continuous and convex function, and the efficient frontier consists of a finite number of hyperbola segments and straight line segments with at most one subinterval corresponding to straight line segments. Moreover, we give some investment strategies by the results we obtained. Finally, we provide a numerical example to illustrate our results.

1. Introduction

The mean–variance model of Markowitz [1,2] is regarded as the cornerstone of modern portfolio theory and leads to a series of theoretical breakthroughs in financial market research. Many mean–variance portfolio selection problems with different constraints are studied. Markowitz [2] and Vörös [3] study the portfolio selection problems with no-shorting constraints. Zhang and Nie [4] further study the case when the expected return and variance of asset have admissible errors. They derive the admissible efficient portfolio in closed form when short sales are not allowed. Pyle and Turnovsky [5] investigate a mean-standard deviation portfolio selection problem with safety-first principle constraint. Alexander et al. [6] study the mean–variance model with Value- at-Risk (**VaR**) and Conditional Value-at-Risk (**CVaR**) constraints when security returns are assumed to have a discrete distribution with finitely many jump points. Xue et al. [7] study a mean–variance portfolio selection problem with concave transaction cost and a series of linear inequality constraints, and design a branch and bound algorithm to solve the problem. Using possibility theory, Chen et al. [8] investigate a possible mean–variance portfolio selection model with transaction costs under upper and lower bound constraints, and propose a cutting plane algorithm to solve their model. Yao and Li [9] analyze the mean–variance model with minimum investment proportion constraint and obtain explicit expressions for the efficient portfolio and the efficient frontier. For more detailed discussion on the subject of portfolio selection problem with constraints, one is referred to Ziemba [10].

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In practice, investors (individuals, fund managers, etc.) always hope that the return of portfolio r_p is greater than or equal to a certain benchmark d_w even in the worst investment condition. This means that the return of portfolio r_p satisfies that the return is at least d_w with probability one, i.e., $\mathbb{P}(r_p \geqslant d_w) = 1$, where $\mathbb{P}$ is a probability measure. A portfolio selection problem with such restriction is referred to as the portfolio investment problem with drawdown constraint. To study such portfolio selection problems is important to both academia and industry.

Some important and interesting results related to this area are briefly summarized as follows. Under the constraint that the loss being no more than a fixed fraction of the maximum wealth achieved at any point in time up to that date, Grossman and Zhou [11] consider a finite-horizon investment-only problem with two securities (one risky and one risk-free) by using continuous-time expected utility maximization model. Cvitanic and Karatzas [12] extend this result to the case when there are multiple risky securities by martingale approach. Elie and Touzi [13] consider infinite-horizon optimal consumption and investment problem, and derive closed-form expressions of the optimal consumption and investment strategy for a general class of utility functions. Elie [14] further investigates the finite-horizon version of Elie and Touzi [13] by viscosity solution approach. Yuan and Hu [15] investigate optimal consumption and portfolio policies with the consumption habit and the terminal wealth downside constraints. Our work differs from those in the literature that we use the mean–variance model rather than continuous-time expected utility maximization model. Chekhlov et al. [16] propose several drawdown measures in portfolio optimization and study the mean-drawdown model and find out the numerical solution to the model when short sales are forbidden. Instead of replacing variance by a drawdown measure as in Chekhlov et al. [16], we add a drawdown constraint to the mean–variance model in this paper.

The closest study in the literature to the current paper is Alexander and Baptista [17]. By using mean–variance and mean-tracking error volatility (TEV) model, Alexander and Baptista [17] analyze a portfolio selection problem with a drawdown constraint in the market environment with finite number of states and derive $K+2$ and $K+3$ fund separation theorems for mean–variance and mean-TEV models, respectively, where K is the number of states at which the constraint binds. However, Alexander and Baptista [17] only study the characterization of efficient portfolios without considering the characterization of the efficient frontier. In this paper, we will not only examine the basic properties of efficient portfolios, but also investigate the composition and geometric feature of the mean–variance efficient frontier with a drawdown constraint. Furthermore, we also provide a Fund Separation Theorem (see Theorem 10 in this paper) in our model, which is consistent with the result of Alexander and Baptista [17]. Therefore, we extend the work of Alexander and Baptista [17] in some sense. Recently, Filomena and Lejeune [18] study a portfolio optimization problem with a similar constraint $\mathbb{P}(r_p \geqslant d_w) \geqslant p$ under normal distribution assumption, where $p \in [0.5, 1)$ is a specified probability level. Our problem differs from that of Filomena and Lejeune [18] in the following two ways. First, under our model, security returns are assumed to have a discrete general distribution rather than normal distribution. Second, their constraint is different from ours which is $\mathbb{P}(r_p \geqslant d_w) = 1$.

Following the setting in Alexander and Baptista [17], we further study the characterizations of efficient frontier for the mean–variance model with a drawdown constraint. Our contributions in this paper are summarized as follows: (1) we prove that the covariance matrix Σ of the securities in the market is singular and obtain a necessary and sufficient condition for the existence of optimal solution to the mean–variance model; (2) we verify that the efficient frontier of the model is a continuous and convex function, and the efficient frontier consists of a finite number of hyperbola segments and straight line segments that are linked together smoothly in the coordinate plane, and with at most one subinterval corresponding to straight line segments; (3) we provide a Fund Separation Theorem; (4) we give an easy implementing scheme based on the obtained results and a numerical example to demonstrate our results.

2. Symbols, concepts and the model

Assume that there are n securities in the financial market. Denote their payoffs, returns and present prices by $z_1, z_2, \ldots, z_n; r_1, r_2, \ldots, r_n$ and $p_1, p_2, \ldots, p_n$, respectively. Then, $r_i = (z_i - p_i)/p_i$, namely, $z_i - p_i = r_i p_i, i = 1, 2, \ldots, n$. Assume that there are m $(\leqslant n)$ future states at the end of the investment period, and the set of the states is denoted by $\Pi = \{1, 2, \ldots, m\}$. Denote the payoff and the return of the ith security at state j by z_{ij} and r_{ij} $(i = 1, \ldots, n; j = 1, \ldots, m)$, respectively. Form vectors $\vec{z}_i = (z_{i1}, z_{i2}, \ldots, z_{im})'$ and $\vec{r}_i = (r_{i1}, r_{i2}, \ldots r_{im})'$[1], $i = 1, \ldots, n$. Denote the mean vector and covariance matrix of returns $r_1, r_2, \ldots, r_n$ by $U = (u_1, u_2, \ldots, u_n)'$ and Σ, respectively; and the portfolio of the securities by $W_N = (w_1, w_2, \ldots, w_n)'$. Let I_k be a k-dimensional vector with each component to be 1 for positive integer k and $R = (\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)_{m \times n}$ be the matrix of returns of the n securities. Let r_p be the return of portfolio with vector form $\vec{r}_p = (r_{p1}, r_{p2}, \ldots, r_{pm})'$ whose expectation and variance to be u_p; and σ_p^2, respectively. Then $\vec{r}_p = \sum_{i=1}^n w_i \vec{r}_i = RW_N$. In this paper, we denote the return of the risk-free security by r_f (> 0). Define

$$span\{\eta_1, \eta_2, \ldots, \eta_n\} = \left\{ \eta_p \middle| \eta_p = \sum_i^n w_i \eta_i : w_1, w_2, \ldots, w_n \in R, \sum_i^n w_i = 1, \right\},$$

i.e., $span\{\eta_1, \eta_2, \ldots, \eta_n\}$ is the overall yield rate space of portfolio with different kinds of returns (or random variable) $\eta_{1,} \eta_{2,} \ldots, \eta_n$. In

[1] We use both random variable form such as z_i or r_i and their vector form such as $\vec{z}_i$ or $\vec{r}_i$ according to the context and actual requirement.