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Convergence analysis for H^1-Galerkin mixed finite element approximation of one nonlinear integro-differential model

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ABSTRACT

In this paper we investigate H^1-Galerkin mixed finite element approximation of one nonlinear integro-differential equation. This method possesses some advantages such as approximating the unknown function and its gradient simultaneously as well as the finite element spaces being free of LBB condition. A priori error estimates of the unknown function and its gradient are derived for both semi-discrete and fully discrete schemes. A numerical example is presented to illustrate the theoretical findings.

1. Introduction

In this paper we are concerned with H^1-Galerkin mixed finite element approximation of the following nonlinear integro-differential model

$$\frac{\partial u}{\partial t} = (1+\sigma(t))\frac{\partial^2 u}{\partial x^2} + f(x,t), (x,t) \in I \times [0,T], \tag{1.1}$$

with the initial and boundary conditions:

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T \tag{1.2}$$

and

$$u(x,0) = u_0(x), \quad x \in I, \tag{1.3}$$

where $I = [0,1],\ \sigma(t) = \int_0^t \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx d\tau$. $u_0(x)$ and $f(x,t)$ are given functions.

This kind of equations often arise in mathematical modeling of the process of a magnetic field penetrating into a substance, see, for example, [12,13], etc. In the past years many finite difference schemes were developed for this type of models. One can refer to [14–16]. Recently Galerkin finite element approximation of model (1.1) was studied in [11], where only a priori error estimate in H^1 norm was derived.

Note that the coefficient in (1.1) depends on the derivative of u. The standard finite difference or finite element methods solve u directly, and then differentiate it to determine the coefficient. Therefore, the resulting coefficient is often inaccurate, which then reduces the accuracy of the numerical approximation for u.

In order to avoid this problem, in this paper we apply the H^1 Galerkin mixed finite element method to solve model (1.1), which can approximate the unknown function u and its derivative simultaneously. The H^1-Galerkin mixed finite element method was proposed in [1] for parabolic problems, which can be viewed as a non-symmetric version of least square

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method. Compared with standard mixed finite element method ([4–7]) the H^1-Galerkin mixed finite element method does not require the finite element spaces to satisfy the LBB stability condition, which makes the choice of finite element spaces more flexible. It has been proved that the H^1-Galerkin mixed finite element method has the same rate of convergence as standard mixed finite element method. For more applications of this mixed formulation, one can refer to [1–3,8,9].

By introducing $q = u_x$ we split model (1.1) into a system of two equations. Applying the H^1-Galerkin mixed finite element method to the resulting systems semi-discrete and fully discrete H^1-Galerkin mixed finite element schemes are defined, respectively. A priori error estimates for u and q are deduced for both semi-discrete scheme and fully discrete scheme. Moreover, a numerical example is given to illustrate the theoretical analysis.

Throughout the paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev space on domain Ω with a norm $\|\cdot\|_{m,q}$ and a semi-norm $|\cdot|_{m,q}$. For $q = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and for $m = 0$, we denote $\|\cdot\| = \|\cdot\|_0$. Moreover, the inner products in $L^2(\Omega)$ are indicated by $(\cdot,\cdot)$. Let X be a Banach space and $\phi(t) : [0,T] \mapsto X$, we set

$$\|\phi\|^2_{L^2(X)} = \int_0^T \|\phi(s)\|^2_X ds, \quad \|\phi\|_{L^\infty(X)} = ess \sup_{0 \leqslant t \leqslant T} \|\phi\|_X.$$

In addition, C denotes a generic constant independent of the spatial mesh parameter h and time discretization parameter $\triangle t$, and ϵ denotes an arbitrarily small positive constant.

The remainder of this article is organized as follows: In Section 2 a semi-discrete H^1-Galerkin mixed finite element scheme is proposed and optimal a priori error estimates are obtained. In Section 3 a completely discrete scheme is briefly described and optimal a priori error bounds are derived for this case. In Section 4 we present a numerical example to illustrate the theoretical results derived in Sections 2 and 3.

2. Semi-discrete H^1-Galerkin mixed finite element formulation

In this section we propose a semi-discrete H^1-Galerkin mixed finite element scheme for (1.1)–(1.3) and derive a priori error estimates.

To define the H^1-Galerkin mixed finite element procedure, we split (1.1) into a system of two equations by introducing $q = u_x$:

$$\begin{cases} q \ = u_x, \\ u_t = (1 + \widetilde{\sigma}(t))q_x + f(x,t), \end{cases} \tag{2.1}$$

where $\widetilde{\sigma}(t) = \int_0^t \int_0^1 q^2 dx d\tau$. Let $H_0^1 = \{v \in H^1(I) | v(0) = v(1) = 0\}$. Multiplying the first equation in (2.1) by $v_x,\ v \in H_0^1$, and integrating on interval I leads to

$$(u_x, v_x) = (q, v_x), \ \ v \in H_0^1.$$

Multiplying the second equation in (2.1) by $w_x,\ w \in H^1$, and integrating on interval I yields

$$(u_t, w_x) - ((1 + \widetilde{\sigma}(t))q_x, w_x) = (f, w_x), \ w \in H^1.$$

Note that $u_t(0,t) = u_t(1,t) = 0$. Integrating on interval I leads to

$$(q_t, w) + ((1 + \widetilde{\sigma}(t))q_x, w_x) + (f, w_x) = 0, \ w \in H^1.$$

Then the weak formulation of (2.1) is to find $\{u,q\} : [0,T] \mapsto H_0^1 \times H^1$ such that

$$\begin{cases} (u_x, v_x) = (q, v_x), \quad v \in H_0^1, \\ (q_t, w) + ((1 + \widetilde{\sigma}(t))q_x, w_x) + (f, w_x) = 0, \quad w \in H^1. \end{cases} \tag{2.2}$$

Let $V_h,\ W_h$ be finite dimensional subspaces of H_0^1 and H^1, respectively, with the following approximation properties:

$$\inf_{v_h \in V_h} \{\|v - v_h\|_{0,p} + h\|v - v_h\|_{1,p}\} \leqslant Ch^{k+1}\|v\|_{k+1,p}, \quad v \in H_0^1 \cap W^{k+1,p}(I)$$

and

$$\inf_{w_h \in W_h} \{\|w - w_h\|_{0,p} + h\|w - w_h\|_{1,p}\} \leqslant Ch^{r+1}\|w\|_{r+1,p}, \quad w \in W^{r+1,p}(I),$$

where $1 \leqslant p \leqslant \infty,\ k,\ r$ are integers.

Then the semi-discrete H^1-Galerkin mixed finite element approximation of (2.2) can be characterized as finding $\{u_h, q_h\} : [0,T] \mapsto V_h \times W_h$ such that

$$\begin{cases} (u_{hx}, v_{hx}) = (q_h, v_{hx}), \quad v_h \in V_h, \\ (q_{ht}, w_h) + ((1 + \widetilde{\sigma}_h(t))q_{hx}, w_{hx}) + (f, w_{hx}) = 0, \quad w_h \in W_h, \end{cases} \tag{2.3}$$

with given $q_h(0)$ and $\widetilde{\sigma}_h(t) = \int_0^t \int_0^1 q_h^2 dx d\tau$.