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Some nonlinear dynamic integral inequalities on time scales

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ABSTRACT

In this paper, we study some dynamic integral inequalities with mixed nonlinearities on time scales, which provide explicit bounds on unknown functions. Our results include many existing ones in the literature as special cases and can be used as tools in the qualitative theory of certain classes of dynamic equations with mixed nonlinearities on time scales.

1. Introduction

Following Hilger's landmark paper [1], there have been plenty of references focused on the theory of time scales in order to unify continuous and discrete analysis, where a time scale is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0, \mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{H}_n$ the space of the harmonic numbers. For the notions used below we refer to [2] that provides some basic facts on time scale.

Recently, many authors have extended some continuous and discrete integral inequalities to arbitrary time scales. For example, see [3–23] and the references cited therein. The purpose of this paper is to further improve and generalize some integral inequalities on time scales that have been studied in a recent papers [9]. By studying integral inequalities mixed nonlinearities on time scales, we first generalize a basic inequality that plays a fundamental role in the proofs of the main results in [9]. Then, we provide improved bounds on unknown functions that can be used as tools in the qualitative theory of certain classes of dynamic equations on time scales.

2. Main results

In this paper, we consider the following nonlinear dynamic integral inequalities

$$x(t) \leqslant a(t) + b(t) \int_{t_0}^{t} [g(s)x(s) + h_1(s)x^{\lambda_1}(\sigma(s)) - h_2(s)x^{\lambda_2}(\sigma(s))]\Delta s, \quad t \in \mathbb{T}^{\kappa} \tag{I}$$

and

$$x(t) \leqslant a(t) + b(t) \int_{t_0}^{t} w(t,s)[g(s)x(s) + h_1(s)x^{\lambda_1}(\sigma(s)) - h_2(s)x^{\lambda_2}(\sigma(s))]\Delta s, \quad t \in \mathbb{T}^{\kappa}, \tag{II}$$

where $a, b, g, h_1, h_2, x : \mathbb{T}^{\kappa} \to \mathbb{R}_+ = [0, \infty)$ are rd-continuous functions, $w : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}_+$ is a continuous function.

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Before stating our main results, we need the following lemmas.

Lemma 1 [2, Th. 6.1, p. 255]. *Let $y, q \in C_{rd}$ and $p \in \mathcal{R}_+(\mathbb{T}, \mathbb{R})$. Then*

$$y^\Delta(t) \leqslant p(t)y(t) + q(t), \quad t \in \mathbb{T}$$

implies

$$y(t) \leqslant y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))q(s)\Delta s, \quad t \in \mathbb{T}.$$

Lemma 2. *Let x be a nonnegative function, $0 < \lambda_1 < 1 < \lambda_2, c_1 \geqslant 0,\ k_2 \geqslant 0, c_2 > 0$ and $k_1 > 0$. Then, for $i = 1, 2$,*

$$(-1)^{i+1}c_i x^{\lambda_i} + (-1)^i k_i x \leqslant \theta_i(\lambda_i, c_i, k_i),$$

where $\theta_i(\lambda_i, c_i, k_i) := (-1)^i(\lambda_i - 1)\lambda_i^{\frac{\lambda_i}{1-\lambda_i}} c_i^{\frac{1}{1-\lambda_i}} k_i^{\frac{\lambda_i}{\lambda_i - 1}}$.

Proof. Set $F_i(x) = (-1)^{i+1}c_i x^{\lambda_i} + (-1)^i k_i x$. It is easy to see that $F_i(x)$ obtains its maximum at $x = \left(\frac{\lambda_i c_i}{k_i}\right)^{\frac{1}{1-\lambda_i}}$ and

$$(F_i)_{\max} = (-1)^i(\lambda_i - 1)\lambda_i^{\frac{\lambda_i}{1-\lambda_i}} c_i^{\frac{1}{1-\lambda_i}} k_i^{\frac{\lambda_i}{\lambda_i - 1}}, \quad \text{for } i = 1, 2. \qquad \square$$

Lemma 3 [2, Th. 1.117, p. 46]. *Suppose that for each $\epsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|w(\sigma(t), \tau) - w(s, \tau) - w_1^\Delta(t, \tau)(\sigma(t) - s)| \leqslant \epsilon|\sigma(t) - s|, \quad s \in U, \tag{2}$$

where $w : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}_+$ is continuous at $(t, t), t \in \mathbb{T}^\kappa$ with $t > t_0$, and $w_1^\Delta(t, \cdot)$ (the derivative of w with respect to the first variable) is rd-continuous on $[t_0, \sigma(t)]$. Then

$$v(t) := \int_{t_0}^{t} w(t, \tau)\Delta\tau$$

implies

$$v^\Delta(t) = \int_{t_0}^{t} w_1^\Delta(t, \tau)\Delta\tau + w(\sigma(t), t).$$

We are now ready to state and prove the main results of this paper.

Theorem 1. *Assume that $a, b, g, h_1, h_2, x : \mathbb{T}^\kappa \to \mathbb{R}_+$ are rd-continuous functions. Then, for any rd-continuous functions $k_1(t) > 0$ and $k_2(t) \geqslant 0$ on $\mathbb{T}^\kappa$ satisfying $k(t) := k_1(t) - k_2(t) \geqslant 0$ and $\mu(t)k(t)b(\sigma(t)) < 1$ for $t \in \mathbb{T}^\kappa$, the inequality (I) implies*

$$x(t) \leqslant a(t) + b(t)\int_{t_0}^{t} e_{A\oplus B}(t, \sigma(s))D(s)\Delta s, \quad t \in \mathbb{T}^\kappa, \tag{3}$$

where

$$A(t) := g(t)b(t), \quad B(t) := \frac{k(t)b(\sigma(t))}{1 - \mu(t)k(t)b(\sigma(t))},$$

$$D(t) := [1 + \mu(t)B(t)]C(t)$$

and

$$C(t) := g(t)a(t) + k(t)a(\sigma(t)) + \theta_1(\lambda_1, h_1, k_1) + \theta_2(\lambda_2, h_2, k_2).$$

Proof. Let

$$y(t) := \int_{t_0}^{t} [g(s)x(s) + h_1(s)x^{\lambda_1}(\sigma(s)) - h_2(s)x^{\lambda_2}(\sigma(s))]\Delta s, \quad t \in \mathbb{T}^\kappa.$$

Then, $y(t_0) = 0$ and (I) can be rewritten as

$$x(t) \leqslant a(t) + b(t)y(t), \quad t \in \mathbb{T}^\kappa. \tag{4}$$