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A combined approach using coupled reduced alternating group explicit (CRAGE) algorithm and sixth order off-step discretization for the solution of two point nonlinear boundary value problems

R.K. Mohanty *, Jyoti Talwar

Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110 007, India

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ABSTRACT

The aim of this paper is to discuss the application of two parameter coupled reduced alternating group explicit (CRAGE) and Newton–CRAGE iteration method to the sixth order accurate numerical solution of the two point singular boundary value problem $-u'' + \phi(x,u,u') = 0$, subject to natural boundary conditions. For all approximations for u and its derivatives, we use only three uniform grid points and for known variable x we use three uniform grid points and two off-step grid points. We also discuss the application of CRAGE iteration method to the singular problem. The convergence of the CRAGE iteration method is discussed in detail. We have compared the results of proposed CRAGE iteration method with the results of corresponding two parameter alternating group explicit (TAGE) iteration methods to demonstrate computationally the superiority of the proposed method.

1. Introduction

Consider the general second order non-linear ordinary differential equation

$$-u'' + \phi(x,u,u') = 0, \quad 0 < x < 1 \tag{1}$$

subject to essential boundary conditions

$$u(0) = A, \quad u(1) = B, \tag{2}$$

where A, B are finite constants.

The two point boundary value problem mentioned above has a unique solution (see Keller [1]). These types of boundary value problems for ordinary differential equations arise very frequently in many complex mathematical modeling problems in science and engineering in which the solution of a linear or non-linear singular ordinary differential equation is required, where the boundary conditions are given at two different points, one of them being a singular point. It is very difficult to achieve an analytical solution to such problems and it becomes necessary to resort to numerical techniques. There are a variety of numerical techniques that can be applied. In 1985, Evans [2] has developed group explicit methods for solving large linear systems, which are suitable for use on parallel computers. Alternating group explicit method, which is a powerful iterative method to solve nonlinear singular two point boundary value problems has been studied by Evans and Mohanty [3], Mohanty et al. [4–6]. In 1990, Evans [7] has introduced alternating group explicit method in a single sweep for the solution of

* Corresponding author.
E-mail addresses: rmohanty@maths.du.ac.in, mohantyranjankumar@gmail.com (R.K. Mohanty), chhabrajyoti@gmail.com (J. Talwar).

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parabolic equation with periodic boundary conditions. These methods are explicit in nature and are suitable for use on parallel computers.

Here, we discuss the sixth order accurate finite difference method which uses three uniform grid points for all approximations for u and its derivatives and three uniform grid points and two off-step grid points for the known variable x (see Chawla [8], Mohanty et al. [9,10] and Jha [11]). The method is also applicable to problems in polar coordinates. In Section 2, we give the description of the method. In Section 3, we discuss the application of the sixth order method to linear singular equation and the Burgers equation. In Section 4, we discuss the coupled reduced alternating group explicit (CRAGE) and Newton–CRAGE iterative method for solving the difference equations obtained on applying the sixth order method to linear and nonlinear problems respectively. Further, in this section, we also discuss the convergence of the CRAGE iterative method in details. In Section 5, we solve two problems and compare the performance of the proposed CRAGE and Newton–CRAGE iterative methods with the corresponding TAGE and Newton–TAGE iterative methods. Concluding remarks are given in Section 6.

2. Sixth order approximation

Consider the uniform mesh spacing $h > 0$ along the x-axis. We introduce a finite set of grid points $x_k = kh,\ k = 0(1)N+1$ with $(N+1)h = 1$, N being a positive integer.

Let the exact solution value of u at the grid point x_k be denoted as $U_k = u(x_k)$. Let u_k be the approximate value of U_k at the grid point x_k.

The method is as follows: Let

$$\bar{U}'_k = (U_{k+1} - U_{k-1})/(2h) \tag{3.1}$$

$$\bar{U}'_{k+1} = (3U_{k+1} - 4U_k + U_{k-1})/(2h) \tag{3.2}$$

$$\bar{U}'_{k-1} = (-3U_{k-1} + 4U_k - U_{k+1})/(2h) \tag{3.3}$$

$$\bar{\phi}_k = \phi(x_k, U_k, \bar{U}'_k) \tag{3.4}$$

$$\bar{\phi}_{k+1} = \phi(x_{k+1}, U_{k+1}, \bar{U}'_{k+1}) \tag{3.5}$$

$$\bar{\phi}_{k-1} = \phi(x_{k-1}, U_{k-1}, \bar{U}'_{k-1}) \tag{3.6}$$

$$\hat{U}'_{k+1} = \bar{U}'_k + \frac{h}{3}(2\bar{\phi}_k + \bar{\phi}_{k+1}) \tag{3.7}$$

$$\hat{U}'_{k-1} = \bar{U}'_k - \frac{h}{3}(2\bar{\phi}_k + \bar{\phi}_{k-1}) \tag{3.8}$$

$$\hat{\phi}_{k+1} = \phi\left(x_{k+1}, U_{k+1}, \hat{U}'_{k+1}\right) \tag{3.9}$$

$$\hat{\phi}_{k-1} = \phi\left(x_{k-1}, U_{k-1}, \hat{U}'_{k-1}\right) \tag{3.10}$$

$$\hat{U}_{k+1/2} = \frac{1}{32}(15U_{k+1} + 18U_k - U_{k-1}) - \frac{h^2}{64}(3\bar{\phi}_{k+1} + 4\bar{\phi}_k - \bar{\phi}_{k-1}) \tag{3.11}$$

$$\hat{U}_{k-1/2} = \frac{1}{32}(-U_{k+1} + 18U_k + 15U_{k-1}) - \frac{h^2}{64}(3\bar{\phi}_{k-1} + 4\bar{\phi}_k - \bar{\phi}_{k+1}) \tag{3.12}$$

$$\hat{U}'_{k+1/2} = \frac{1}{4h}(5U_{k+1} - 6U_k + U_{k-1}) - \frac{h}{48}(3\bar{\phi}_{k+1} + 8\bar{\phi}_k + \bar{\phi}_{k-1}) \tag{3.13}$$

$$\hat{U}'_{k-1/2} = \frac{1}{4h}(-U_{k+1} + 6U_k - 5U_{k-1}) + \frac{h}{48}(\bar{\phi}_{k+1} + 8\bar{\phi}_k + 3\bar{\phi}_{k-1}) \tag{3.14}$$

$$\hat{\phi}_{k+1/2} = \phi\left(x_{k+1/2}, \hat{U}_{k+1/2}, \hat{U}'_{k+1/2}\right) \tag{3.15}$$

$$\hat{\phi}_{k-1/2} = \phi\left(x_{k-1/2}, \hat{U}_{k-1/2}, \hat{U}'_{k-1/2}\right) \tag{3.16}$$

$$\hat{\hat{U}}'_k = \bar{U}'_k + \frac{h}{156}\left[2(\bar{\phi}_{k+1} - \bar{\phi}_{k-1}) - 3\left(\hat{\phi}_{k+1} - \hat{\phi}_{k-1}\right) - 24\left(\hat{\phi}_{k+1/2} - \hat{\phi}_{k-1/2}\right)\right] \tag{3.17}$$

$$\hat{\hat{\phi}}_k = \phi\left(x_k, U_k, \hat{\hat{U}}'_k\right) \tag{3.18}$$

Then the sixth order finite difference method (see Chawla [8]) for the proposed differential equation (1) is given by

$$U_{k+1} - 2U_k + U_{k-1} = \frac{h^2}{60}\left[\hat{\phi}_{k+1} + \hat{\phi}_{k-1} + 16\left(\hat{\phi}_{k+1/2} + \hat{\phi}_{k-1/2}\right) + 26\hat{\hat{\phi}}_k\right] + O(h^8), \quad k = 1(1)N \tag{4}$$

where $U_0 = u_0 = A$ and $U_{N+1} = u_{N+1} = B$.