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# Essential norm of products of multiplication composition and differentiation operators on weighted Bergman spaces

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ABSTRACT

Let $\psi$ be a holomorphic function on the open unit disk $\mathbb{D}$ and $\varphi$ a holomorphic self-map of $\mathbb{D}$. Let $C_\varphi, M_\psi$ and $D$ denote the composition, multiplication and differentiation operator, respectively. We find an asymptotic expression for the essential norm of products of these operators on weighted Bergman spaces on the unit disk. This paper is a continuation of our recent paper concerning the boundedness of these operators on weighted Bergman spaces.


## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the space of all functions holomorphic on $\mathbb{D}$ and $H^\infty(\mathbb{D}) = H^\infty$ the space of all bounded holomorphic functions with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. Let $dm(z) = \frac{1}{\pi}dxdy$ be the normalized area measure on $\mathbb{D}$ (i.e. $m(\mathbb{D}) = 1$). For each $\alpha \in (-1,\infty)$, we set

$$dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z), \quad z \in \mathbb{D}.$$

Let $\mathcal{L}^p_\alpha(\mathbb{D}) = \mathcal{L}^p_\alpha$, $p > 0$, $\alpha > -1$, be the weighted Lebesgue space containing all measurable functions $f$ on $\mathbb{D}$ such that $\int_\mathbb{D} |f(z)|^p dm_\alpha(z) < \infty$. By $A^p_\alpha(\mathbb{D}) = A^p_\alpha$ we denote the space $\mathcal{L}^p_\alpha(\mathbb{D}) \cap H(\mathbb{D})$, which is called the weighted Bergman space. For $1 \leqslant p < \infty$ the space is Banach with the norm

$$\|f\|_{A^p_\alpha} = \left(\int_\mathbb{D} |f(z)|^p dm_\alpha(z)\right)^{1/p}.$$

For the case $\alpha = 0$, space $A^p_\alpha$ will be denoted simply by $A^p$. It is well known that $f \in A^p_\alpha$ if and only if $f'(z)(1-|z|^2) \in \mathcal{L}^p_\alpha$. Moreover the following asymptotic relation holds

$$\|f\|^p_{A^p_\alpha} \asymp |f(0)|^p + \int_\mathbb{D} |f'(z)|^p (1-|z|^2)^p dm_\alpha(z). \tag{1}$$

Let $\varphi$ be a holomorphic self-map on $\mathbb{D}$. The composition operator $C_\varphi$ induced by $\varphi$ is defined by $(C_\varphi f)(z) = (f\circ\varphi)(z)$, $f \in H(\mathbb{D})$. For $\psi \in H(\mathbb{D})$ the multiplication operator $M_\psi$ is defined on $H(\mathbb{D})$ by $M_\psi f(z) = \psi(z)f(z), f \in H(\mathbb{D})$. The differentiation operator denoted by $D$ is defined by $Df = f', f \in H(\mathbb{D})$. Some results on products of concrete linear operators can be found, e.g. in [4–11,16–18,21–49] (see also the references therein).

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The products of composition, multiplication and differentiation operators can be defined in the following six ways

$$\begin{aligned}&(M_\psi C_\varphi Df)(z)=\psi(z)f'(\varphi(z));\\&(M_\psi DC_\varphi f)(z)=\psi(z)\varphi'(z)f'(\varphi(z));\\&(C_\varphi M_\psi Df)(z)=\psi(\varphi(z))f'(\varphi(z));\\&(DM_\psi C_\varphi f)(z)=\psi'(z)f(\varphi(z))+\psi(z)\varphi'(z)f'(\varphi(z));\\&(C_\varphi DM_\psi f)(z)=\psi'(\varphi(z))f(\varphi(z))+\psi(\varphi(z))f'(\varphi(z));\\&(DC_\varphi M_\psi f)(z)=\psi'(\varphi(z))\varphi'(z)f(\varphi(z))+\psi(\varphi(z))\varphi'(z)f'(\varphi(z))\end{aligned}\tag{2}$$

for $z\in\mathbb{D}$ and $f\in H(\mathbb{D})$. From operator $M_\psi C_\varphi D$ for $\psi(z)=1$, we get operator $C_\varphi D$, while for $\psi(z)=\varphi'(z)$, we get operator $DC_\varphi$, which have been studied, for example, in [4,6,9,11,14,17,28,30].

To treat operators in (2) in a unified manner, we introduced in [38] the following operator

$$T_{\psi_1,\psi_2,\varphi}f(z)=\psi_1(z)f(\varphi(z))+\psi_2(z)f'(\varphi(z)),\quad f\in H(\mathbb{D}),\tag{3}$$

where $\psi_1,\psi_2\in H(\mathbb{D})$ and $\varphi$ is a holomorphic self-map of $\mathbb{D}$. It is clear that all products of composition, multiplication and differentiation operators in (2) can be obtained from the operator $T_{\psi_1,\psi_2,\varphi}$ by fixing $\psi_1$ and $\psi_2$. Indeed, we have $M_\psi C_\varphi D=T_{0,\psi,\varphi};\ M_\psi DC_\varphi=T_{0,\psi\varphi',\varphi};\ C_\varphi M_\psi D=T_{0,\psi\circ\varphi,\varphi};\ DM_\psi C_\varphi=T_{\psi',\psi\varphi',\varphi};\ C_\varphi DM_\psi=T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi};\ DC_\varphi M_\psi=T_{(\psi'\circ\varphi)\varphi',\ (\psi\circ\varphi)\varphi',\varphi}$.

Let $0<\beta<\infty$. Recall that a positive Borel measure $\mu$ on $\mathbb{D}$ is called a $\beta$-Carleson measure if

$$\|\mu\|_\beta:=\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^\beta}<\infty,$$

where $S(I)=\{z:1-|I|\leqslant|z|<1,z/|z|\in I\}$ is a Carleson box based on the arc $I\subset\partial\mathbb{D}$ of length $|I|>0$. Measure $\mu$ is a vanishing $\beta$-Carleson measure if

$$\lim_{|I|\to0}\frac{\mu(S(I))}{|I|^\beta}=0.$$

This paper is continuation of its part one (see [38]), where we characterized the boundedness of operator (3) on weighted Bergman spaces by proving the following result.

**Theorem 1.** *Let $1\leqslant p<\infty,\alpha\in(-1,\infty),\psi_1,\psi_2\in H(\mathbb{D})$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$.*

(1) *If $\psi_1\in H^\infty$, then the following statements are equivalent:*
 (i) *$T_{\psi_1,\psi_2,\varphi}$ is bounded on $A^p_\alpha$.*
 (ii) *The pull-back measure $\mu_{\psi_2,\varphi,\alpha,p}=v_{\psi_2,\alpha,p}\circ\varphi^{-1}$ of $v_{\psi_2,\alpha,p}$ induced by $\varphi$ is an $(\alpha+2+p)$-Carleson measure.*
 (iii) $\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|z|^2)^{\alpha+2}}{|1-\bar z\varphi(w)|^{2(\alpha+2)+p}}|\psi_2(w)|^p dm_\alpha(w)<\infty.$

(2) *If $\psi_2$ satisfies the condition*

$$M:=\sup_{z\in\mathbb{D}}\frac{|\psi_2(z)|}{1-|\varphi(z)|^2}<\infty,\tag{4}$$

*then the following statements are equivalent:*
 (i) *$T_{\psi_1,\psi_2,\varphi}$ is bounded on $A^p_\alpha$.*
 (ii) *The pull-back measure $\mu_{\psi_1,\varphi,\alpha,p}=v_{\psi_1,\alpha,p}\circ\varphi^{-1}$ of $v_{\psi_1,\alpha,p}$ induced by $\varphi$ is an $(\alpha+2)$-Carleson measure.*
 (iii) $\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|z|^2)^{\alpha+2}}{|1-\bar z\varphi(w)|^{2(\alpha+2)}}|\psi_1(w)|^p dm_\alpha(w)<\infty.$

Here we estimate the essential norm of the operator, and apply these results to concrete operators listed in (2). Recall that the essential norm $\|T\|_e$ of a bounded linear operator $T$ on a Banach space $X$ is given by

$$\|T\|_e=\inf\{\|T+K\|_X:K\quad\text{is compact on }X\},$$

that is, its distance in the operator norm from the space of compact operators on $X$. The essential norm provides a measure of non-compactness of $T$. Clearly, $T$ is compact if and only if $\|T\|_e=0$. For some results in the topic see, e.g. [1–3,15,24,25,31,42], and the related references therein.

Throughout this paper, constants are denoted by $C$, they are positive and not necessarily the same at each occurrence. The notation $A\asymp B$ means that there is a positive constant $C$ such that $B/C\leqslant A\leqslant CB$.