Applied Mathematics and Computation 218 (2011) 2398–2406

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Common fixed points of generalized contractions on partial metric spaces and an application

Ljubomir Ćirić [a,*,1], Bessem Samet [b], Hassen Aydi [c], Calogero Vetro [d]

[a] Faculty of Mechanical Engineering, Kraljice Marije 16, Belgrade, Serbia
[b] Université de Tunis, Ecole Supérieure des Sciences et Techniques de Tunis, 5, Avenue Taha Hussein-Tunis, BP 56, Bab Menara-1008, Tunisia
[c] Université de Monastir, Institut Supérieur d'Informatique de Mahdia, Route de Réjiche, Km 4 BP 35, Mahdia 5121, Tunisia
[d] University of Palermo, Department of Mathematics and Informatics, Via Archirafi 34, 90123 Palermo, Italy

ARTICLE INFO

Keywords:
Coincidence point
Common fixed point
Partial metric space
Weakly compatible pair of mappings
Homotopy

ABSTRACT

In this paper, common fixed point theorems for four mappings satisfying a generalized nonlinear contraction type condition on partial metric spaces are proved. Presented theorems extend the very recent results of I. Altun, F. Sola and H. Simsek [Generalized contractions on partial metric spaces, Topology and its applications 157 (18) (2010) 2778–2785]. As application, some homotopy results for operators on a set endowed with a partial metric are given.

1. Introduction and preliminaries

Partial metric spaces were introduced by Matthews [8] in 1992 as a part of the study of denotational semantics of dataflow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation [6,10,14–16,19].

First, we recall some definitions of partial metric space and some their properties [8–10,13,18].

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

(p1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(p2) $p(x,x) \leqslant p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 1.1. It is clear that, if $p(x,y) = 0$, then from (p1) and (p2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}_+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_+$.

Each partial metric p on X generates a T_0 topology τ_p on Xwhich has as a base the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

* Corresponding author.
E-mail addresses: lciric@rcub.bg.ac.rs (L. Ćirić), bessem.samet@gmail.com (B. Samet), hassen.aydi@isima.rnu.tn (H. Aydi), cvetro@math.unipa.it (C. Vetro).
[1] Author is supported by Grant No.174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.07.005

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}_+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

is a metric on X.

Definition 1.2. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

Definition 1.3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Remark 1.2. It is easy to see that every closed subset of a complete partial metric space is complete.

Lemma 1.1 ([8,9]). *Let (X,p) be a partial metric space. Then*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*
(b) *(X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Matthews [8] obtained the following Banach fixed point theorem on complete partial metric spaces.

Theorem 1.1 [8]. *Let f be a mapping of a complete partial metric space (X,p) into itself such that there is a real number c with $0 \leqslant c < 1$, satisfying for all $x, y \in X$:*

$$p(fx,fy) \leqslant cp(x,y).$$

Then f has a unique fixed point.

Very recently, I. Altun, F. Sola and H. Simsek [2] considered a contraction of Ćirić's type in partial metric spaces and proved the following nice result that generalizes Theorem 1.1 of Matthews. For other works that treat the Ćirić's contraction condition in usual metric spaces, we refer the reader to [3,4,11,12,17].

Theorem 1.2 [2]. *Let (X,p) be a complete partial metric space and let $T:X \to X$ be a map such that*

$$p(Tx,Ty) \leqslant \varphi\left(\max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right\}\right)$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous non-decreasing function such that $\varphi(t) < t$ and the series $\sum_{n\geqslant 1}\varphi^n(t)$ converges for all $t > 0$. Then T has a unique fixed point.

We note that in Theorem 1.2 formal the condition that the series $\sum_{n\geqslant 1}\varphi^n(t)$ converges does not exist, but is used in the proof of Theorem 1.2.

In the present paper, we derive some common fixed point theorems for two pairs of weakly compatible mappings satisfying a generalized contraction condition on partial metric spaces. The presented theorems extend and generalize some results of I. Altun, F. Sola and H. Simsek [2]. As application, we derive some homotopy results for operators on a set endowed with a partial metric.

2. Main results

Before stating the main results, we recall the following definitions.

Definition 2.1. Let X be a non-empty set and $T_1, T_2 : X \to X$ are given self-maps on X. If $w = T_1x = T_2x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2.

Definition 2.2 [7]. Let X be a non-empty set and $T_1, T_2 : X \to X$ are given self-maps on X. The pair $\{T_1,T_2\}$ is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.

Our main result is the following.