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Impulsive nonlocal differential equations through differential equations on time scales

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ABSTRACT

We propose a non-standard approach to impulsive differential equations in Banach spaces by embedding this type of problems into differential (dynamic) problems on time scales. We give an existence result for dynamic equations and, as a consequence, we obtain an existence result for impulsive differential equations.

1. Introduction

In the present paper we study the nonlocal impulsive Cauchy problem in an arbitrary Banach space X:

$$\dot{y}(t) = f(t, y(t)), \quad \text{a.e. } t \in [0, b] \setminus \{t_1, \ldots, t_m\}, \tag{1}$$

$$\Delta y(t_i) = I_i(y(t_i)), \quad \forall i \in \{1, \ldots, m\}, \tag{2}$$

$$y(0) = g(y) + y^0. \tag{3}$$

Here $[0, b] \subset \mathbb{R}$ and $0 < t_1 < \ldots < t_m < b$ are the pre-assigned moments of impulse, $\Delta y(t) = y(t+) - y(t-)$ denotes the "jump" of the function y at t and the discontinuity at the point t_i is described by the function $I_i : X \to X$.

In contrast to the earlier papers, we will show some connections between this theory and a theory of dynamic equations on time scales. In fact, the method of the proof is based on the equivalence of such problems with some differential equations on a constructed time scale $\mathbb{T}$. This idea can be found in the paper of Hilger [32] but he never pursued it any further. However, some impulsive differential equations are considered directly on time scales (cf. [35] or [36], for instance). We disagree with this idea because, considering the requirements on the jumps, they should not be isolated points. The situation is exactly the same as in differential equations on $\mathbb{R}$. We propose to treat the impulsive Cauchy problem as a special case of the dynamic Cauchy problem on time scales. This idea is parallel to the embedding of one dynamics into another one. The converse direction of the embeddability (time scales dynamics in ODE dynamics) was considered by Garay and Hilger [26] and in recent papers of Akhmet and Turan [4,5].

Let us note that the presented idea remains true under an arbitrary set of assumptions which guarantee the existence of solutions for the problems considered. We restrict our attention to the case when $y \mapsto I_i(y(t_i))$ is continuous for each $i \in \{1, \ldots, m\}$, $g : PC([0, b], X) \to X$ is a bounded and continuous mapping and f satisfies a Henstock–Lebesgue integrability assumption as well as a kind of generalized Carathéodory condition.

First, we need to describe the construction of (HL) Δ-integrals. Next we present an idea of construction of a time scale associated with the nonlocal impulsive Cauchy problem. At that point we will be ready to present our existence results.

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Thus we solve two problems: the dynamic nonlocal Cauchy problem in a Banach space (by using the notion of the Henstock–Lebesgue integral on time scales) and the second one, the impulsive nonlocal problem on a usual real interval, as a consequence. Some necessary properties of this new kind of integral are also presented. This integral is known for functions defined on real intervals and has a growing number of interesting applications in differential and integral equations ([24,40,42,41], for instance). We also extend these results by considering the dynamic equations on time scales.

We need to clarify, very briefly, all the aspects of the paper. The nonlocal conditions are often motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [14] or [15]. As indicated in [14] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be more descriptive in physics with better effect than the classical initial condition $y(0) = y_0$. For well over a century, differential equations have been used in forming the dynamics of changing processes. Development of the modeling theory has been accompanied by a comprehensive theory for differential equations.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics whose duration is negligible in comparison with the duration of the entire evolution. It is natural to treat these perturbations acting instantaneously or in the form of "impulses" in the above mentioned models. The theory of impulsive differential equations, as a consequence, has been developed in modeling problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth (see [8,34] or [3]).

Finally, some remarks about the chosen method of the proof. The calculus of time scales was initiated by Stefan Hilger in his Ph.D. thesis in 1988 (supervised by Bernd Aulbach) in order to create a theory that can unify discrete and continuous analysis. The current investigations are devoted not only to unify differential and difference equations, but also q-difference equations and other types of problems. This novel and fascinating type of mathematics is more general and versatile than the traditional theories of differential and difference equations. Under one framework we are able to describe continuous-discrete hybrid processes, hence it is the optimal way for accurate and malleable mathematical modeling. In fact, the field of dynamic equations on time scales contains and extends the classical theory of differential and difference equations (cf. [12,13]). We try to indicate the next interesting field of applications for dynamic problems on time scales: linking the theory of differential (or integral) equations with impulses with that of dynamic equations on time scales.

2. Notations and preliminary facts

To understand the so-called dynamic equation and easily follow this paper we present some preliminary definitions and notations which are very common in the literature (see [12,13] and references therein).

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$.

The three most popular examples of calculi on time scales are the differential calculus, the difference calculus, and the quantum (q-difference) calculus i.e. when $\mathbb{T} = \mathbb{R},\ \mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$ for $q > 1$, respectively.

If a, b are points in $\mathbb{T}$, then we denote by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$ the time scale interval.

Definition 1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

We put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

The jump operators σ and ρ allow the classification of points in a time scale in the following way: t is called *right dense*, *right scattered*, *left dense*, *left scattered*, *dense* and *isolated* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

Let X be a Banach space. Denote by $C(\mathbb{T}, X)$ the space of X-valued continuous functions on $\mathbb{T}$ and by $B_R(x^0)$ its closed ball of radius R centered in a fixed function $x^0 \in X$.

Definition 2. Let $f : \mathbb{T} \to X$ and $t \in \mathbb{T}$. We define Δ-derivative $f^{\Delta}(t)$ as the element of X (if it exists) with the property that for any $\varepsilon > 0$ there exists a neighborhood of t on which

$$\|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\| \leqslant \varepsilon|\sigma(t) - s|.$$

Remark 3. Concerning the Δ-derivative, it turns out that

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,
(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$, and
(iii) $f^{\Delta} = \Delta_q f$ is the q-derivative if $\mathbb{T} = q^{\mathbb{N}_0} = q^t : t \in \mathbb{N}_0$, $q > 1$.

Accordingly, time scales allow us to unify the treatment of differential and difference equations (and not only these ones).

Similarly, one can define the ∇-derivative: