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A filtering method for the interval eigenvalue problem

Milan Hladík [a,b,*], David Daney [b], Elias Tsigaridas [c]

[a] Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800 Prague, Czech Republic
[b] INRIA Sophia-Antipolis Méditerranée, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex, France
[c] Department of Computer Science, Aarhus University, IT-parken, Aabogade 34, DK 8200 Aarhus N, Denmark

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ABSTRACT

We consider the general problem of computing intervals that contain the real eigenvalues of interval matrices. Given an outer approximation (superset) of the real eigenvalue set of an interval matrix, we propose a filtering method that iteratively improves the approximation. Even though our method is based on a sufficient regularity condition, it is very efficient in practice and our experimental results suggest that it improves, in general, significantly the initial outer approximation. The proposed method works for general, as well as for symmetric interval matrices.

1. Introduction

In order to model real-life problems and perform computations we *must* deal with inaccuracy and inexactness; these are due to measurement, to simplification assumption on physical models, to variations of the parameters of the system, and finally due to computational errors. Interval analysis is an efficient and reliable tool that allows us to handle the aforementioned problems, even in the worst case where all together are encountered simultaneously. The input quantities are given with some interval estimation and the algorithms output verified intervals as results that (even though they usually have the drawback of overestimation) cover all the possibilities for the input quantities.

We are interested in the interval real eigenvalue problem. Given a matrix the elements of which are real intervals, also called interval matrix, the task is to compute real intervals that contain all possible eigenvalues. For formal definitions we refer the reader to the next section.

Moreover, there is a need to distinguish general interval matrices from the symmetric ones. Applications arise mostly in the field of mechanics and engineering. We name, for instance, automobile suspension system [1], mass structures [2], vibrating systems [3], robotics [4], and even principal component analysis [5] and independent component analysis [6], which could be considered as a statistics oriented applications. Using the well-known Jordan–Wielandt transformation [7,8], if we are given a solution of the interval real eigenvalue problem, we can provide an approximation for the singular values and the condition number; both quantities have numerous applications.

The first general results for the interval real eigenvalue problem were produced by Deif [9], and Deif and Rohn [10]. However their solutions depend on theorems that have very strong assumptions. Later, Rohn [11], introduced a boundary point characterization of the eigenvalue set. Approximation methods were addressed by Qiu et al. [1], Leng et al. [12] and by Hladík et al. [13]. The works [12,13] are based on a branch and prune approach and yield results that depend on a given arbitrarily high accuracy.

* Corresponding author at: Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800 Prague, Czech Republic.

E-mail addresses: milan.hladik@matfyz.cz, hladik@kam.mff.cuni.cz (M. Hladík), david.daney@sophia.inria.fr (D. Daney), elias.tsigaridas@gmail.com (E. Tsigaridas).

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The symmetric eigenvalue problem is very important in practice. However, in its interval form it is hard to handle with since the correlations between the entries of the matrices force the algorithms of interval analysis to overestimate actually a lot the results. The symmetric case was pioneered by Deif [9]. Another theoretical result is attributed to Hertz [14], see also [15], to determine two extremal points of the eigenvalue set. Diverse approximation algorithms have also been developed. An evolution strategy method by Yuan et al. [16] gives an inner approximation (subset) of the eigenvalues set, and under some conditions it converges to the exact bounds. Matrix perturbation theory was used by Qiu et al. [2], who proposed an algorithm for approximating the bounds, and by Leng and He [17] for outer approximation (superset) of the eigenvalue set. Outer bounds that are easy and fast to compute were presented by Hladík et al. [18], and the general parametric case was considered by Kolev [19].

In this paper, we propose a filtering method to reduce the overestimation produced by various methods. Generally, filtering methods start with an initial outer approximation and iteratively make it tighter. Even though filtering is commonly used approach in constrained programming, it is not widely used for the interval eigenvalue problem. We can, of course, apply any filtering for the interval nonlinear system of equations arising from eigenvalue definition, but no such approach has been successful yet; cf. [13]. To the best of our knowledge, the only related work is by Beaumont [20], where an iterative algorithm is presented, based on convex approximation of eigenpairs. The new filtering method that we propose is more simple and applicable for both the symmetric and unsymmetric cases. Since we do not take into account eigenvectors, it is much more efficient, too.

2. Basic definitions and main theorem

An interval matrix is defined as a family of matrices

$$\boldsymbol{A} := [\underline{A},\overline{A}] = \{A \in \mathbb{R}^{m\times n},\ \underline{A} \leqslant A \leqslant \overline{A}\},$$

where $\underline{A},\ \overline{A} \in \mathbb{R}^{m\times n},\ \underline{A} \leqslant \overline{A}$, are given matrices, and the inequality is considered element-wise. By $A_c := \frac{1}{2}(\underline{A}+\overline{A})$, and $A_\Delta := \frac{1}{2}(\overline{A}-\underline{A})$ we denote the midpoint and the radius of $\boldsymbol{A}$, respectively.

Let $\boldsymbol{A} \subseteq \mathbb{R}^{n\times n}$ be a square interval matrix. Its eigenvalue set is defined as

$$\Lambda(\boldsymbol{A}) := \{\lambda \in \mathbb{R}; Ax = \lambda x, x \neq 0, A \in \boldsymbol{A}\}.$$

That is, matrices in $\boldsymbol{A}$ may have both real and complex eigenvalues, but we focus on the real ones only. An outer approximation of $\Lambda(\boldsymbol{A})$ is any set having $\Lambda(\boldsymbol{A})$ as a subset. An important class of matrices is that of symmetric ones. Its generalization to interval matrices is as follows. A *symmetric interval matrix* is defined as

$$\boldsymbol{A}^S := \{A \in \boldsymbol{A} | A = A^T\}.$$

Without loss of generality assume that $\boldsymbol{A}^S$ is non-empty, which is easy to check, and that A_Δ and A_c are symmetric. Its eigenvalue set is denoted similarly to generic case, that is

$$\Lambda(\boldsymbol{A}^S) := \{\lambda \in \mathbb{R};\ Ax = \lambda x,\ x \neq 0,\ A \in \boldsymbol{A}^S\}.$$

Since $\boldsymbol{A}^S$ is a proper subset of $\boldsymbol{A}$, its eigenvalue set, $\Lambda(\boldsymbol{A}^S)$, is in general a subset of $\Lambda(\boldsymbol{A})$.

Since a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ has always n real eigenvalues, we can sort them in a non-increasing order as follows

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A).$$

We extend this notation for symmetric interval matrices, that is $\boldsymbol{\lambda}_i(\boldsymbol{A}^S) := \left\{\lambda_i(A) | A \in \boldsymbol{A}^S\right\}$. These sets form n convex compact intervals, which can be disjoint or may overlap, see for example [18]. The union of these interval results in $\Lambda(\boldsymbol{A}^S)$. We denote their outer approximations by $\boldsymbol{\omega}_i(\boldsymbol{A}^S) \supseteq \boldsymbol{\lambda}_i(\boldsymbol{A}^S)$, where $i = 1,\ldots,n$.

Let $\rho(\cdot)$ be the spectral radius, and $|\cdot|$ the matrix absolute value, understood componentwise. We now present our main theoretical result, which employs the sufficient regularity conditions by Beeck [21] and Rump [22] (compare Rex and Rohn [23]), and allows us to present the filtering method.

Theorem 1. *Let $\lambda^0 \notin \Lambda(\boldsymbol{A})$ and define $\mathbf{M} := \boldsymbol{A} - \lambda^0 I$. Then $(\lambda^0 + \lambda) \notin \Lambda(\boldsymbol{A})$ for all real λ satisfying*

$$|\lambda| < \frac{1 - \frac{1}{2}\rho\left(|I - QM_c| + |I - QM_c|^T + |Q|M_\Delta + M_\Delta^T|Q|^T\right)}{\frac{1}{2}\rho\left(|Q| + |Q|^T\right)}, \tag{1}$$

where $Q \in \mathbb{R}^{n\times n},\ Q \neq 0$, is an arbitrary matrix.

Proof. It is sufficient to prove that every λ satisfying (1) the interval matrix $\mathbf{M} - \lambda I = \boldsymbol{A} - \lambda^0 I - \lambda I$ is regular, i.e., consists of nonsingular matrices only.