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Application of homotopy perturbation methods for solving systems of linear equations

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A R T I C L E I N F O

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A B S T R A C T

In this paper, homotopy perturbation methods (HPMs) are applied to obtain the solution of linear systems, and conditions are deduced to check the convergence of the homotopy series. Moreover, we have adapted the Richardson method, the Jacobi method, and the Gauss–Seidel method to choose the splitting matrix. The numerical results indicate that the homotopy series converges much more rapidly than the direct methods for large sparse linear systems with a small spectrum radius.

1. Introduction

Many numerical methods, such as iterative methods [5] and HPMs [4], are suggested to search for the solution of linear systems. Keramati [4] applied a HPM to solve linear systems. The splitting matrix of this method is only the identity matrix. However, this method does not converge for some systems when the spectrum radius is greater than 1. To make the method available, the auxiliary parameter and the auxiliary matrix are added to the homotopy method; in other words, the HPMs are applied to consider the linear systems. Nowadays, HPMs are utilized to determine the solution of many problems, such as nonlinear heat transfer equations [2], nonlinear wave equations [2,3,4] and van der Pol equations [6].

In addition to HPMs, many analytic techniques have been proposed using homotopy, a fundamental concept of topology. The idea of homotopy is a continuous map from the interval $[0,1]$ to a function space. To solve a nonlinear equation $f(u) = 0$, the continuation procedure is done to track the set of zeros of

$$H(u,p) = (1-p)f(u) + pg(u) = 0,$$

where $g(u) = 0$ is a starting problem whose solution u_0 is known. Watson [7] pointed out that continuation can fail because the curve of zeros of $H(u,p)$ emanating from $(u_0, 0)$ may (i) have turning points, (ii) bifurcate, (iii) fail to exist at some p values, or (iv) wander off to infinity without reaching $p = 1$. In 1976, Chow et al. [1] proposed the probability-one homotopies that was able to overcome all the four shortcomings of the continuation and homotopy methods.

In this study, the homotopy was always continuous for linear systems. The goal of this paper was to propose a new iterative method for solving large sparse linear systems

$$Ax = b, \tag{1}$$

where

$$A = [a_{ij}], \quad x = [x_j], \quad b = [b_j], \quad i = 1,2,\ldots,n, \; j = 1,2,\ldots,n.$$

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Instead of using the traditional homotopy, HPMs introduce a nonzero auxiliary parameter and a nonzero auxiliary matrix. Hence, the advantage of HPM is the freedom to choose the auxiliary parameter, where the auxiliary parameters may play the same role as overrelaxation parameters in iteration methods [5].

In summary, the main contributions of this paper are

(1) to give an approximation for (1) by applying HPMs,
(2) to derive conditions for checking the convergence of the homotopy series,
(3) to adapt the Richardson method, Jacobi method and Gauss–Seidel method for choosing the splitting matrix.

The numerical studies show that the homotopy series converges rapidly when the spectrum radius of the linear system is small.

This paper is organized as follows. In Section 2, we have introduced the basic concept of HPM, and have derived the conditions for the convergence of the homotopy series. In Section 3, we have introduced the homotopy Richardson method, homotopy Jacobi method, and homotopy Gauss–Seidel method. In Section 4, we have presented a numerical example which is a large sparse linear system. The concise conclusion is provided in Section 5.

2. Basic concept in HPMs

We have considered the HPMs in a general mathematical setting. A general type of homotopy method for solving (1) can be described as follows.

By applying a certain matrix Q, we can define homotopy $H(u,p)$ by

$$H(u,0) = F(u) \text{ and } H(u,1) = L(u)$$

and a convex homotopy as follows

$$H(u,p) = (1-p)F(u) + (qH)pL(u) = 0, \tag{2}$$

where

$$L(u) = Au - b \text{ and } F(u) = Qu - w_0.$$

Here, we have the freedom to choose the auxiliary parameter q, the auxiliary matrix H, the initial approximation w_0, and the auxiliary operator $F(u)$. Note that the operator $F(u)$ is decided by the splitting matrix Q.

HPM uses the homotopy parameter p as an expanding parameter to obtain

$$u = u_0 + u_1 p + u_2 p^2 + \cdots \tag{3}$$

and it gives an approximation to the solution of (1) as

$$v = \lim_{p\to 1}(u_0 + u_1 p + u_2 p^2 + \cdots).$$

By substituting (3) in (2), and by equating the terms with the identical power of p, we can obtain

$$\begin{aligned}
&p^0 : Qu_0 - w_0 = 0\\
&p^1 : Qu_1 + (qHA - Q)u_0 + w_0 - qHb = 0\\
&p^i : Qu_i + (qHA - Q)u_{i-1} = 0,\ i = 2,3,\ldots
\end{aligned}$$

This implies

$$\begin{aligned}
&u_0 = Q^{-1}w_0\\
&u_1 = (I - qQ^{-1}HA)u_0 + Q^{-1}(qHb - w_0)\\
&u_i = (I - qQ^{-1}HA)u_{i-1},\ i = 2,3,\ldots
\end{aligned}$$

Taking $w_0 = qHb$ yields

$$\begin{aligned}
&u_0 = q(Q^{-1}H)b\\
&u_i = (I - qQ^{-1}HA)^i(qQ^{-1}H)b,\ i = 1,2,3,\ldots
\end{aligned} \tag{4}$$

Therefore,

$$u = \sum_{i=0}^{\infty} u_i p^i = \sum_{i=0}^{\infty}\left[(I - qQ^{-1}HA)^i(qQ^{-1}H)b\right]p^i. \tag{5}$$