Applied Mathematics and Computation 217 (2011) 5500–5507

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Sixth order derivative free family of iterative methods

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ARTICLE INFO

Keywords:
Iterative methods
Non-linear equations
Derivative free
Convergence order
Newton

ABSTRACT

In this study, we develop a four-parameter family of sixth order convergent iterative methods for solving nonlinear scalar equations. Methods of the family require evaluation of four functions per iteration. These methods are totally free of derivatives. Convergence analysis shows that the family is sixth order convergent, which is also verified through the numerical work. Though the methods are independent of derivatives, computational results demonstrate that family of methods are efficient and demonstrate equal or better performance as compared with other six order methods, and the classical Newton method.

1. Introduction

A wide class of problems which arises in various disciplines of mathematical and engineering sciences can be formulated in terms of nonlinear equations of the form

$$f(x) = 0, \tag{1}$$

[4]. One of the best known and probably the most used method for solving the above equation is Newton's method (**NM**). The classical Newton method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, \ldots. \tag{2}$$

NM converges quadratically in some neighborhood of the solution γ. In recent years, much attention have been given to develop and analyze a number of numerical methods for solving the nonlinear equations (see [1–18] and references therein). These methods require evaluation of derivatives. It is possible that the computational cost of evaluating these derivatives is very high [12–16,22,23]. Motivated by optimization considerations, we develop iterative methods which are totally free of derivatives and are sixth order convergent. Let us now review various recently developed sixth order convergent iterative methods.

The sixth order convergent methods presented in [6–9,11] consist of three steps. Based upon the well known Jarrat's method [3] recently Ren et al. [7,11] proposed the following sixth order convergent iterative family of methods consisting of three-steps and using two-parameters (**RWB**)

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{(2a-b)f'(x_n)+bf'(y_n)+cf(x_n)}{(-a-b)f'(x_n)+(3a+b)f'(y_n)+cf(x_n)}\frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{3}$$

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doi:10.1016/j.amc.2010.12.021

where $a, b, c \in \mathbf{R}$ and $a \neq 0$. Wang et al. [7] also developed a six order convergent variant of the Jarrat's method. Their method consists of three-steps and uses two-parameters (**WKL**)

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{(5\alpha+3\beta)f'(x_n)-(3\alpha+\beta)f'(y_n)}{2\alpha f'(x_n)+2\beta f'(y_n)}\frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{4}$$

where $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta \neq 0$. Lately Sharma and Guha [8] modified the Ostrowski method [1] and developed the following sixth order convergent method consisting of three steps and using one parameter (**SG**)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)+af(y_n)}{f(x_n)+(a-2)f(y_n)}\frac{f(z_n)}{f'(x_n)}, \end{cases} \tag{5}$$

where $a \in \mathbf{R}$. Earlier, Neta [9] has developed the sixth order convergent family of methods consisting of three steps and using one parameter (**NETA**)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)+af(y_n)}{f(x_n)+(a-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)-f(y_n)}{f(x_n)-3f(y_n)}\frac{f(z_n)}{f'(x_n)}. \end{cases} \tag{6}$$

We notice that, in the preceding method, the choice $a = -1$ produces the same correcting factor in the last two steps. Chun and Ham [6] also developed a sixth order modification of the Ostrowski's method. Their family of methods consists of the following three steps (**CH**)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(u_n)\frac{f(z_n)}{f(x_n)}, \end{cases} \tag{7}$$

where $u_n = f(y_n)/f(x_n)$, and $H(t)$ represents a real valued function satisfying $H(0) = 1$, $H'(0) = 2$. In the case

$$H(t) = \frac{1+(\beta+2)t}{1+\beta t}, \tag{8}$$

the third substep is similar to method of Sharma and Guha [8].

We notice that the methods (3) and 4) require evaluation of two derivatives and two functions while the methods (5)–(7) require evaluation of one derivative and three functions during each iterative step. The efficiency index of an iterative method is given as: $\xi^{1/m}$ [6–8,17]. Here, ξ is the convergence order of the method and m is the number of function evaluations. If we assume that the computational cost of evaluating a function is same as evaluating its derivative. Then, the efficiency index of the methods (3)–(7) is $6^{1/4}$. Derivating functions and evaluating them may be computationally expensive. For example, evaluating derivative of $f(x) = x^k$ with $k < 0$ or $f(x) = \exp(\cos(x))$ is more expansive than evaluating $f(x)$. We develop family of methods which are totally free of derivatives and require evaluation of only four functions at each iterative step. Therefore the efficiency index of our methods is also $6^{1/4}$ which is better than the efficiency index of **NM**. Our methods are not only totally free of derivatives but are also efficient as demonstrated through the computational results.

2. The methods and convergence analysis

Let us consider a three step iterative scheme in the following form so, we can develop methods free of derivatives

$$\begin{aligned} y_n &= x_n - \kappa\frac{f(x_n)^2}{f(x_n)-f(x_n-\kappa f(x_n))}, \\ z_n &= y_n - \kappa\frac{f(x_n)f(y_n)}{f(x_n)-f(x_n-\kappa f(x_n))}[1 + Aw_1(x_n,y_n) + Bw_1(x_n,y_n)^2 + Cw_2(x_n,y_n) + Dw_2(x_n,y_n)^2], \\ x_{n+1} &= z_n - \kappa\frac{f(x_n)f(z_n)}{f(x_n)-f(x_n-\kappa f(x_n))}[1 + Ew_1(x_n,y_n) + Fw_1(x_n,y_n)^2 + Gw_2(x_n,y_n) + Hw_2(x_n,y_n)^2 + Iw_3(y_n,z_n)], \end{aligned}$$