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On the $p(x)$-Laplacian Robin eigenvalue problem ☆

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ABSTRACT

Consider Robin eigenvalue problem involving the $p(x)$-Laplacian on a smooth bounded domain Ω as follows:

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial\gamma} + \beta(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of infinitely many eigenvalue sequences if $p(x) \not\equiv$ constant and also present some sufficient conditions for which there is no principal eigenvalue and the set of all eigenvalues is not closed.

1. Introduction

Stimulated by the development of the study of elastic mechanics, electrorheological fluids and image restoration (see [26,27,23,4]), interest in variational problems and differential equations with variable exponent has grown in recent decades. For example, see [6–9,11,13,14] and references therein and refer to [17] for an overview of this subject.

The $p(x)$-Laplacian Dirichlet, Neumann, and Steklov eigenvalue problems on a bounded domain have been investigated and some interesting results have been obtained (see [14,9,5] and references therein). We would like to mention Prof. Xianling Fan who has made contributions to the $p(x)$-Laplacian eigenvalue problems (see [14,9,10]). The authors, Mihăilescu, Rădulescu and Stancu-Dumitru, discuss the different aspects of eigenvalue problems with variable exponents (see [21,22] and references therein). The investigations mainly have relied on variational methods and deduce the existence of infinitely many eigenvalue sequences and also give some sufficient conditions for which the infimum of all eigenvalues is zero or positive, respectively.

For the p-Laplacian Dirichlet, Neumann, No-flux, Robin and Steklov eigenvalue problems, it is well known that:

(1) There exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ tending to ∞ as $n \to \infty$ (see [2,20]);
(2) There exists a principal eigenvalue λ_1, which is the smallest of all possible eigenvalues, as a consequence of minimization results of appropriate functionals (see [20]);
(3) The set of eigenvalues is closed (see [1,20]);

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(4) The first eigenvalue λ_1 is isolated (see [19,20]);
(5) The eigenvalue λ_2 is the second eigenvalue (see [1,20]), i.e.

$$\lambda_2 = \inf\{\lambda : \lambda \text{ is an eigenvalue of the problem and } \lambda > \lambda_1\}.$$

In fact, excepting the case when $p = 2$, the set of eigenvalues for the p-Laplace operator is not completely described in the sense that excepting the eigenvalues given by the Ljusternik–Schnirelmann method the possible existence of other eigenvalues in the interval (λ_1, ∞) is an open question in higher dimensions.

The operator $-\Delta_{p(x)}u = -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ with $p(x) > 1$ is called the $p(x)$-Laplacian which is a natural generalization of the p-Laplacian (where $p > 1$ is a constant). When $p(x) \not\equiv$ constant, the $p(x)$-Laplacian possesses more complicated nonlinearity than the p-Laplacian, say, it is nonhomogeneous. For this reason, some of the above properties of the p-Laplacian eigenvalue problems may not hold for a general $p(x)$-Laplacian case. We have obtained the following properties, which are different from the p-Laplacian case:

(1) For the $p(x)$-Laplacian eigenvalue problems with $p(x) \not\equiv$ constant, there exist infinitely many eigenvalue sequences $\{\lambda_{(n,\alpha)}\}$ tending to ∞ as $n \to \infty$ (see Section 3 and [14,9,5]);
(2) In [14,5] and Section 4, for the $p(x)$-Laplacian Dirichlet, Steklov and Robin eigenvalue problems, under some assumptions on $p(x)$ the infimum of all eigenvalues is zero but zero is not an eigenvalue. This means that for some variable exponent $p(x)$ there is no principal eigenvalue and the set of eigenvalues is not closed. Even if there exists a principal eigenvalue λ_*, this λ_* may not be isolated because λ_* is the infimum of all eigenvalues.
(3) For the $p(x)$-Laplacian Neumann eigenvalue problem, the smallest eigenvalue of the problem, λ_1, is zero and simple, all eigenfunctions associated with λ_1 are nonzero constant functions, but under some assumptions on $p(x)$ the first eigenvalue, λ_1, is not isolated, namely, the infimum of all positive eigenvalues of the problem is zero. It means that there is no second eigenvalue for some variable exponent $p(x)$ (see [9]).

The purpose of the present paper is to study the following Robin eigenvalue problem involving the $p(x)$-Laplacian which is a new topic

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \gamma} + \beta(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{R}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \gamma}$ denotes the outer normal derivative of u with respect to $\partial\Omega$, $p \in C(\overline{\Omega}, \mathbb{R})$ with $p(x) > 1$, $\lambda \in \mathbb{R}$, $\beta : \partial\Omega \to \mathbb{R} \in L^\infty(\partial\Omega)$ is a real function with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$. We prove that there exist infinitely many eigenvalue sequences for (R) if $p(x) \not\equiv$ constant (see Section 3) and also present some sufficient conditions for which the infimum of all eigenvalues of (R) is zero (see Section 4). In order to obtain these results, in Section 2, we state some elementary properties of the space $W^{1,p(x)}(\Omega)$.

2. Variable exponent Sobolev space $W^{1,p(x)}(\Omega)$

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $p \in C(\overline{\Omega}, \mathbb{R})$ with $p(x) > 1$. Then $1 < p^- := \inf_{x\in\Omega} p(x) \leqslant p^+ := \sup_{x\in\Omega} p(x) < +\infty$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{u|\ u : \Omega \to \mathbb{R} \text{ is a real measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}$$

with the norm

$$|u|_{p(x)} = |u|_{L^{p(x)}(\Omega)} = \inf\left\{\tau > 0 : \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leqslant 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = \inf\left\{\tau > 0 : \int_\Omega \left[\left|\frac{\nabla u}{\tau}\right|^{p(x)} + \left|\frac{u}{\tau}\right|^{p(x)}\right] dx \leqslant 1\right\}.$$