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Solving nonlinear equations by a new derivative free iterative method ☆

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ABSTRACT

We develop a new simple iteration formula, which does not require any derivatives of $f(x)$, for solving a nonlinear equation $f(x) = 0$. It is proved that the convergence order of the new method is quadratic. Furthermore, the new method can approximate complex roots. By several numerical examples we show that the presented method will give desirable approximation to the root without a particularly good initial approximation and be efficient for all cases, regardless of the behavior of $f(x)$.

1. Introduction

We consider a root finding problem for a nonlinear equation $f(x) = 0$. In recent years many iterative methods have been proposed [1–15]. These methods are mostly based on the well-known Newton's method. When an initial approximation is not properly chosen or the function $f(x)$ behaves pathologically near the root, however, approximate roots obtained by the iterative methods tend to converge very slowly or even fail to converge. In addition, derivatives of $f(x)$ are necessary in most higher order iterative methods.

To overcome the necessity for derivatives of $f(x)$ and the problem of choosing an initial approximation, the author has recently introduced a simple iterative method [16]. Convergence order of this method was shown to be quadratic, but yet the convergence rate is more or less slow for some particular cases.

In this paper we develop a new derivative free iterative method. The resultant iteration formula can be used in finding complex roots, and it appears to be similar with that of Müller's method whose convergence order is about 1.84 [17]. However, it is proved that the presented iterative method converges quadratically. In the last section, results of several numerical examples show availability and superiority of the presented method over existing iterative methods such as Newton's method and Müller's method.

2. Development of a new iterative method

Suppose that a function $f(x)$ is continuous on a given interval $[a,b]$ and let x_k be an approximation to a root p of an equation $f(x) = 0$. Set $a_k := x_k - h_k$ and $b_k := x_k + h_k$ for some $h_k > 0$. Referring to the idea proposed in [16], we develop an iteration formula as below:

We denote by $Q(x_k;x)$ and $Q(p;x)$ two quadratic polynomials interpolating $f(x)$ at the points $x = a_k, x_k, b_k$ and $x = a_k, p, b_k$, respectively. These polynomials can be represented by

$$Q(x_k;x) := \frac{(x-x_k)(x-a_k)}{(b_k-x_k)(b_k-a_k)}f(b_k) + \frac{(x-b_k)(x-a_k)}{(x_k-b_k)(x_k-a_k)}f(x_k) + \frac{(x-x_k)(x-b_k)}{(a_k-x_k)(a_k-b_k)}f(a_k) \tag{1}$$

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and

$$Q(p;x) := \frac{(x-p)(x-a_k)}{(b_k-p)(b_k-a_k)}f(b_k) + \frac{(x-p)(x-b_k)}{(a_k-p)(a_k-b_k)}f(a_k). \tag{2}$$

We define two integrals as follows.

$$I_k := \int_{a_k}^{b_k} Q(x_k;x)dx = \frac{b_k-a_k}{6}\{f(a_k)+4f(x_k)+f(b_k)\} \tag{3}$$

and

$$I_p := \int_{a_k}^{b_k} Q(p;x)dx = \frac{b_k-a_k}{6(b_k-p)(p-a_k)}\{(p-a_k)(a_k+2b_k-3p)f(b_k)-(b_k-p)(2a_k+b_k-3p)f(a_k)\}. \tag{4}$$

If we set $p = x_{k+1}$ in (4) and take an equation $I_k = I_p$ then it follows that

$$(b_k-x_{k+1})(x_{k+1}-a_k)\{f(a_k)+4f(x_k)+f(b_k)\} = (x_{k+1}-a_k)(a_k+2b_k-3x_{k+1}t)f(b_k)-(b_k-x_{k+1})(2a_k+b_k-3x_{k+1})f(a_k).$$

Solving this equation for x_{k+1}, with $(a_k + b_k)/2 = x_k$, we have

$$x_{k+1} = x_k - \frac{b_k-a_k}{4[f(a_k)+f(b_k)-2f(x_k)]}\left\{f(b_k)-f(a_k)\pm\sqrt{D_k}\right\}, \tag{5}$$

where

$$D_k = \{f(b_k)-f(a_k)\}^2 - 8f(x_k)\{f(a_k)+f(b_k)-2f(x_k)\} \tag{6}$$

and

$$a_k = x_k - h_k, \quad b_k = x_k + h_k, \quad k \geqslant 0. \tag{7}$$

Therein we define the radius h_k of the interval $[a_k,b_k]$ by

$$h_k = |x_k - x_{k-1}|, \quad k \geqslant 1 \tag{8}$$

with initial values $h_0 = (b - a)/2$ and $x_0 = (a + b)/2$.

Taking account of the loss of significance errors in numerical performance, we rationalize the numerator in (5) to obtain

$$x_{k+1} = x_k - \frac{2(b_k-a_k)f(x_k)}{f(b_k)-f(a_k)\pm\sqrt{D_k}}, \quad k \geqslant 0. \tag{9}$$

In this formula the sign is chosen to maximize the magnitude of the denominator. Equivalently, the sign in the formula (5) is chosen to minimize the magnitude of the numerator. In addition, it should be noted that the presented method can approximate complex roots when $D_k < 0$.

The formula (9) seems to be rather simple, compared to the well-known Müller's method as below.

$$x_{k+1} = x_k - \frac{2C_k}{B_k \pm \sqrt{B_k^2 - 4A_kC_k}}, \quad k \geqslant 0, \tag{10}$$

where

$$\begin{aligned}
C_k &= f(x_k),\\
B_k &= \frac{(x_{k-2}-k_k)^2[f(x_{k-1})-f(x_k)]-(x_{k-1}-k_k)^2[f(x_{k-2})-f(x_k)]}{(x_{k-2}-x_k)(x_{k-1}-x_k)(x_{k-2}-x_{k-1})},\\
A_k &= \frac{(x_{k-1}-k_k)[f(x_{k-2})-f(x_k)]-(x_{k-2}-k_k)[f(x_{k-1})-f(x_k)]}{(x_{k-2}-x_k)(x_{k-1}-x_k)(x_{k-2}-x_{k-1})}
\end{aligned}$$

with initial values $x_{-2} = a$, $x_{-1} = (a + b)/2$ and $x_0 = b$. It is well known that the convergence order of the Müller's method is about 1.84. However, the new method (5) or (9) has a quadratic convergence order as shown in the next section.

For numerical implementation of the presented method in (9) we provide the following algorithm.

Algorithm

Step 1. Set initial values:
$h := (b - a)/2$
$x := (a + b)/2$
Step 2. For an integer $K > 0$ and a tolerance error $\epsilon > 0$ given, perform the iterations (formulas (6)–(9)):
$k := 0$