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Kunita–Watanabe inequalities and Tanaka formula for multi-dimensional G-Brownian motion

Defei Zhang [a,b,*], Ping He [b], Xinsong Yang [b]

[a] Department of Mathematics, Shandong University, Jinan 250100, China
[b] Department of Mathematics, Honghe University, Mengzi 661100, China

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ABSTRACT

In this paper, some new properties and interesting estimations of mutual variation process for G-Brownian motion are presented, Kunita–Watanabe inequalities and Tanaka formula for multi-dimensional G-Brownian motion are also obtained.

1. Introduction

Motivated by the risk measure, superpricing and model uncertainty in finance, Peng introduced G-normal distribution under the framework of sublinear expectation space in 2006 (see [1]). With this G-normal distribution, G-Brownian motion and related calculus of Itô's type were given. Unlike the Brownian motion in the classical sense, the G-Brownian motion is not based on a classical probability space. G-Brownian motion has a very rich and interesting new structure which non-trivial generalizes the classical one. Peng in [2–5] introduced the notion of quadratic and mutual variation process for G-Brownian motion and gave some results about quadratic variation process for one and multi-dimensional G-Brownian motion. Since then, there are many studies on G-Brownian motion, for example, [6–10], etc. However, there are few results on mutual variation process for multi-dimensional G-Brownian motion. In this paper we will further study the properties of mutual variation process, we give some new properties and interesting estimations for mutual variation process, the well known Kunita–Watanabe inequalities (see [11]) for G-Brownian motion are also obtained. Of course, many properties of quadratic variation process are the special case of our results. Lin (see [9]) obtained Tanaka formula for one dimensional G-Brownian motion, we extend it to multi-dimensional case.

This paper is organized as follows: in Section 2, we recall briefly some notions and properties about G-expectation and G-Brownian motion. In Section 3, some properties and interesting estimations for mutual variation process are presented. In Section 4, the Tanaka formula for multi-dimensional G-Brownian motion is obtained and its proof is given.

2. Preliminaries

In this section, we introduce some notations and preliminaries about the theory of sublinear expectations and G-Brownian motion, which will be needed in what follows. More details of this section can be founded in [1,2,6].

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued bounded functions defined on Ω. We suppose that $\mathcal{H}$ satisfies $C \in \mathcal{H}$ for each constant C and $|X| \in \mathcal{H}$, if $X \in \mathcal{H}$.

* Corresponding author at: Department of Mathematics, Shandong University, Jinan 250100, China.
E-mail addresses: zhdefei@163.com (D. Zhang), hepingky@163.com (P. He), xinsongyang@163.com (X. Yang).

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Definition 2.1. A sublinear expectation $\mathbb{E}$ is a functional $\mathbb{E}:\mathcal{H}\to\mathcal{R}$ satisfying

(i) Monotonicity: $\mathbb{E}[X]\geqslant\mathbb{E}[Y]$ if $X\geqslant Y$.
(ii) Constant preserving: $\mathbb{E}[C]=C$ for $C\in R$.
(iii) Sub-additivity: For each $X,Y\in\mathcal{H},\mathbb{E}[X+Y]\leqslant\mathbb{E}[X]+\mathbb{E}[Y]$.
(iv) Positive homogeneity: $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X]$ for $\lambda\geqslant 0$.

The triple $(\Omega,\mathcal{H},\mathbb{E})$ is called a sublinear expectation space. If (i) and (ii) are satisfied, $\mathbb{E}$ is called a nonlinear expectation and the triple $(\Omega,\mathcal{H},\mathbb{E})$ is called a nonlinear expectation space.

Remark 2.2. The notion of sublinear expectations is proved to be a basic tool in volatility uncertainty which is crucial in superhedging, superpricing (see [12]) and measures of risk in finance which caused a great attention in finance since the pioneer work of Artzner, Delbaen, Eber and Heath [13,14].

We introduce the notion of G-normal distribution. We denote by $lip(R^d)$ the space of all bounded and Lipschitz real functions on R^d. A G-normal distribution is a nonlinear expectation defined on $lip(R^d)$

$$\mathbb{E}[\varphi]=u(1,0):\varphi\in lip(R^d)\to R,$$

where $u(t,x)$ is a bounded continuous function on $[0,\infty)\times R^d$ which is the viscosity solution of the following nonlinear parabolic partial differential equation:

$$\frac{\partial u}{\partial t}-G(D^2u)=0,\quad u(0,x)=\varphi(x),(t,x)\in[0,\infty)\times R^d,$$

where $G(A)=\frac{1}{2}\sup_{\gamma\in\Gamma}tr[\gamma\gamma^TA],A\in S(d)$. $S(d)$ denotes the space of $d\times d$ symmetric matrices. Γ is a given non-empty, bounded and closed subset of $R^{d\times d}$, the space of all $d\times d$ matrices.

We denote by $\Omega=C_0^d(R^+)$ the space of all R^d-valued continuous paths $(\omega_t)_{t\in R^+}$, with $\omega_0=0$, equipped with the distance

$$\rho(\omega^1,\omega^2):=\sum_{i=1}^{\infty}2^{-i}\left[\left(\max_{t\in[0,i]}|\omega_t^1-\omega_t^2|\right)\wedge 1\right].$$

$\boldsymbol{\Omega}$ is the classical canonical space and $B_t(\omega)=(\omega_t)_{t\geqslant 0}$ is the corresponding canonical process. For each $T>0$, set

$$L_{ip}^0(\mathcal{H}_T):=\left\{\varphi(B_{t_1},B_{t_2},\ldots,B_{t_m}):n\geqslant 1,t_1,\ldots,t_m\in[0,T],\varphi\in lip\left(R^{d\times m}\right)\right\},$$

define

$$L_{ip}^0(\mathcal{H}):=\bigcup_{n=1}^{\infty}L_{ip}^0(\mathcal{H}_n).$$

We set $B_t(\omega)=\omega_t,t\in[0,\infty)$ for $\omega\in\Omega$.

Definition 2.3. The canonical process B_t is called a (d-dimensional) G-Brownian motion under the expectation $\mathbb{E}$ defined on $L_{ip}^0(\mathcal{H})$ if

(i) For each $s,t\geqslant 0$ and $\varphi\in lip(R^d)$, B_t and $B_{t+s}-B_s$ are identically distributed:

$$\mathbb{E}[\varphi(B_{t+s}-B_s)]=\mathbb{E}[\varphi(B_t)].$$

(ii) For each $m=1,2,\ldots,0\leqslant t_1<\cdots<t_m<\infty$, the increment $B_{t_m}-B_{t_{m-1}}$ is backwardly independent from $B_{t_1},\ldots,B_{t_m}$ in the following sense: for each $\phi\in lip(R^{d\times m})$,

$$\mathbb{E}[\phi(B_{t_1},\ldots,B_{t_{m-1}},B_{t_m})]=\mathbb{E}[\phi_1(B_{t_1},\ldots,B_{t_{m-1}})],$$

where $\phi_1(x_1,\ldots,x_{m-1})=\mathbb{E}[\phi(x_1,\ldots,x_{m-1},B_{t_m}-B_{t_{m-1}}+x_{m-1})],\ x_1,\ldots,x_{m-1}\in R^d$.

(iii)The related conditional expectation of $\phi(B_{t_1},\ldots,B_{t_{m-1}},B_{t_m})$ under $\mathcal{H}_{t_k}$ is defined by

$$\mathbb{E}[\phi(B_{t_1},\ldots,B_{t_k},\ldots,B_{t_m})|\mathcal{H}_{t_k}]=\phi_{m-k}(B_{t_1},\ldots,B_{t_k}),$$

where

$$\phi_{m-k}(x_1,\ldots,x_k)=\mathbb{E}[\phi(x_1,\ldots,x_k,B_{t_{k+1}}-B_{t_k}+x_k,\ldots,B_{t_m}-B_{t_k}+x_k)].$$

We denote by $L_G^p(\mathcal{H}_t),p\geqslant 1$, the completion of $L_{ip}^0(\mathcal{H}_t)$ under the norm $\|X\|_p:=(\mathbb{E}[|X|^p])^{1/p}$. Similarly, denote $L_G^p(\mathcal{H})$ is complete space of $L_{ip}^0(\mathcal{H})$. The expectation $\mathbb{E}[\cdot]:L_G^1(\mathcal{H})\to R$ introduced through the above procedure is called G-expectation.