Applied Mathematics and Computation 217 (2011) 5889–5894

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the logarithmic coefficients of Bazilevič functions

Qin Deng

School of Science, Hangzhou Dianzi University, Hangzhou 310018, PR China

ARTICLE INFO

Keywords:
Univalent functions
Bazilevič functions
Logarithmic coefficients

ABSTRACT

The objective of the present paper is to study the logarithmic coefficients of Bazilevič functions. We obtain the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for Bazilevič functions.

1. Introduction

Throughout the paper, $\mathscr{A}$ denotes the class of analytic functions $f(z)$ in the unit disk $U = \{z \in C : |z| < 1\}$ normalized so that $f(0) = 0$ and $f'(0) = 1$.

Let α and β be real numbers with $\alpha > 0$. A function $f(z) \in \mathscr{A}$ is called Bazilevič functions of type (α, β) if

$$f(z) = \left[(\alpha + i\beta)\int_0^z p(t)(g(t))^{\alpha} t^{i\beta - 1} dt\right]^{\frac{1}{\alpha + i\beta}}, \tag{1.1}$$

for a starlike (univalent) function $g(z)$ in U and an analytic function $p(z)$ with $p(0) = 1$ satisfying $Re\{p(z)\} > 0$ in U. We denote by $B(\alpha, \beta)$ the class of Bazilevič functions of type (α, β). For the sake of brevity we shall simply denote by $B(\alpha)$ instead of $B(\alpha, 0)$ and we shall call a function in $B(\alpha)$ a Bazilevič functions of type α.

Let $S, \mathscr{K}, S^*$ and S_c denote the subclasses of $\mathscr{A}$ of functions univalent, convex, starlike and close-to-convex, respectively. We also denote by $\mathscr{P}$ the class of analytic functions $p(z)$ with $p(0) = 1$ and $Re\{p(z)\} > 0$ in U. Note that $\mathscr{P}$ is known as the Carathéodory class.

Let $\alpha > 0$ and $\beta \in R$. In view of (1.1), for $f(z) \in \mathscr{A}$, we readily see that $f(z) \in B(\alpha, \beta)$ if and only if

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} \in \mathscr{P}, \tag{1.2}$$

for some $g(z) \in S^*$ (see [1]).

Bazilevič [2] shows that $B(\alpha, \beta) \subset S$ for $\alpha > 0$, $\beta \in R$. Later, Sheil-Small [3] extends it to the case $\alpha \geqslant 0$ and gives a geometric characterization for $B(\alpha, \beta)$. So far, Bazilevič functions form the largest known subclass of S which has concrete expressions.

It is well known that the inclusion relations

$$\mathscr{K} \subset S^* \subset S_c \subset B(\alpha) \subset B(\alpha, \beta) \subset S$$

are valid. See [2,4,5] for further information.

Associated with each $f(z)$ in S is a well defined logarithmic function

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n, \quad z \in U. \tag{1.3}$$

The numbers γ_n are called the logarithmic coefficients of $f(z)$. Thus the Koebe function $k(z) = z(1 - z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$. It is clear that $|\gamma_1| \leqslant 1$ for each $f(z) \in S$. The problem of the best upper bounds for $|\gamma_n|$ is still open. In fact even

E-mail address: Dqsx123@126.com

0096-3003/$ - see front matter
doi:10.1016/j.amc.2010.12.075

the proper order of magnitude is still not known. It is known, however, for the starlike functions that the best bound is $|\gamma_n| \leqslant \frac{1}{n}$ and that this is not true in general [4, P.151]; [6, P.898]; [7, P.140] and [8].

In the paper [9] it is pointed out that the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for circularly symmetric functions.

In a recent paper [10], it is presented that the inequality $|\gamma_n| \leqslant \frac{1}{n}$ holds also for close-to-convex functions. However, it is pointed out in [11] that there are some errors in the proof and, hence, the result is not substantiated. It is proved in [12] that there exists a function $f(z) \in S_c$ such that $|\gamma_n| > \frac{1}{n}$. Furthermore, it is proved in [13] that the inequality $|\gamma_n| \leqslant An^{-1}\log n$ holds for close-to-convex functions, where A is an absolute constant.

In the present paper, we study the logarithmic coefficients of Bazilevič functions $B(\alpha,\beta)$. Also, we obtain the inequality $|\gamma_n| \leqslant An^{-1}\log n$ (A is an absolute constant) which holds for Bazilevič functions $B(\alpha,\beta)$.

2. Main results

First, we give the following lemmas:

Lemma 1 [13]. *Let $f(z) \in S$. Then, for $z = re^{i\theta}$, $\frac{1}{2} \leqslant r < 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta \leqslant 1 + \frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}, \tag{2.1}$$

and

$$\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2 d\theta dr \leqslant 1 + 2log\frac{1}{1-r}. \tag{2.2}$$

Lemma 2. *Let $f(z) \in S$, $\tau \in C$. Then, $z = re^{i\theta}$, $0 < r < 1$,*

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) = \tau\frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right). \tag{2.3}$$

Proof. It is clear that

$$\frac{zf'(z)}{f(z)} = \frac{1}{i}\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right) + 1. \tag{2.4}$$

It follows that

$$Re\frac{zf'(z)}{f(z)} = Im\left\{\frac{\partial}{\partial\theta}\left(\log\frac{f(z)}{z}\right)\right\} + 1 = \frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right) + 1. \tag{2.5}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{1}{i\tau}\frac{\partial}{\partial\theta}\left(\log\left(\frac{f(z)}{z}\right)^{\tau}\right) + 1, \tag{2.6}$$

then

$$Re\frac{zf'(z)}{f(z)} = \frac{1}{\tau}Im\left\{\frac{\partial}{\partial\theta}\left(\log\left(\frac{f(z)}{z}\right)^{\tau}\right)\right\} + 1 = \frac{1}{\tau}\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) + 1. \tag{2.7}$$

From (2.5) and (2.7) we obtain

$$\frac{\partial}{\partial\theta}\left(arg\left(\frac{f(z)}{z}\right)^{\tau}\right) = \tau\frac{\partial}{\partial\theta}\left(arg\frac{f(z)}{z}\right). \qquad \square$$

Theorem 1. *Let $f(z) \in B(\alpha,\beta)$. Then, for $n \geqslant 2$,*

$$|\gamma_n| \leqslant An^{-1}\log n, \tag{2.8}$$

where A is an absolute constant, and the exponent -1 is the best possible.

Proof. If $f(z) \in B(\alpha,\beta)$, then there exist $g(z) \in S^*$ such that $Re\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta} > 0, \alpha > 0, \beta \in R$. Write $p(z) = \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{i\beta}$, then $Rep(z) > 0$. It is clear that

$$p(z) = 2Rep(z) - \overline{p(z)}.$$