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Fibonacci identities via the determinant of tridiagonal matrix

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ABSTRACT

In this paper, using the method of Laplace expansion to evaluate the determinant tridiagonal matrices, we construct a kind of determinants to give new proof of the Fibonacci identities.

1. Introduction

The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geqslant 1$

$$F_{n+1} = F_n + F_{n-1}, \tag{1.1}$$

with $F_0 = 0$, $F_1 = 1$. There is a long tradition of using matrices and determinants to study Fibonacci numbers. For example, Bicknell–Johnson and Spears [1] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers. Cahill and Narayan [2] show how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. Strang [3], Ayse Nalli [4] and Haci Civciv [5] present a family of tridiagonal matrices given by

$$M(n) = \begin{bmatrix} 3 & 1 & & & & \\ 1 & 3 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 3 & 1 \\ & & & & 1 & 3 \end{bmatrix}_{n\times n}, \tag{1.2}$$

it is easy to show by induction that the determinant $|M(n)|$ is the Fibonacci number F_{2n+2}. Another example in [6,7] is the family of tridiagonal matrices given by:

$$H(n) = \begin{bmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{bmatrix}_{n\times n}, \tag{1.3}$$

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where i is the imaginary unit, i.e. $i^2 = -1$. Strang [8] presents the tridiagonal matrix

$$D_n = \begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & 1 \end{bmatrix}_{n\times n}, \quad (1.4)$$

the determinants $|H(n)|$ and $|D_n|$ are the Fibonacci numbers F_{n+1}. In [4], the authors propose a generalization of symmetric tridiagonal family of matrices, whose determinants form any linear subsequence of the Fibonacci numbers.

In this paper, we utilize the method of Laplace expansion to evaluate the determinant $|H(n)|$ or $|D_n|$ and construct a kind of 2×2 determinants to give two new methods to proof the following Fibonacci identity [9]:

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n. \quad (1.5)$$

We can rewrite (1.5) in another form:

$$F_n = F_kF_{n-k+1} + F_{k-1}F_{n-k}, \quad (1.6)$$

where k is an arbitrary integer, such as $1 \leqslant k \leqslant n$. We have

$$\begin{aligned} F_n &= F_1F_n + F_0F_{n-1}; \\ F_n &= F_2F_{n-1} + F_1F_{n-2}; \\ F_n &= F_3F_{n-2} + F_2F_{n-3}; \\ &\cdots \\ F_n &= F_{n-1}F_2 + F_{n-2}F_1; \\ F_n &= F_nF_1 + F_{n-1}F_0. \end{aligned} \quad (1.7)$$

2. Using the method of Laplace expansion to proof

In 1772, Pierre–Simon Laplace presented the following generalized version of the cofactor. For an $n\times n$ matrix A, let $A([i_1,i_2,\cdots i_k],[j_1j_2,\cdots j_k])$ be the $k\times k$ submatrix of A that lies on the intersection of rows $i_1,i_2,\cdots i_k$ with columns $j_1j_2,\cdots j_k$. And let $M([i_1,i_2,\cdots i_k],[j_1j_2,\cdots j_k])$ be the $(n-k)\times(n-k)$ minor determinant obtained by deleting rows $i_1,i_2,\cdots i_k$ and columns $j_1j_2,\cdots j_k$ from A. The cofactor of $A([i_1,i_2,\cdots i_k],[j_1j_2,\cdots j_k])$ is defined to be the signed minor

$$\mathring{A}([i_1,i_2,\cdots i_k],[j_1,j_2,\cdots j_k]) = (-1)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k}M([i_1,i_2,\cdots i_k],[j_1,j_2,\cdots j_k]). \quad (2.1)$$

For each fixed set of row indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$,

$$|A| = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} |A([i_1,i_2,\cdots i_k],[j_1,j_2,\cdots j_k])|\cdot\left|\mathring{A}([i_1,i_2,\cdots i_k],[j_1,j_2,\cdots j_k])\right|. \quad (2.2)$$

Especially, if $A(i,j) = a_{ij}$, then $\mathring{A}(i,j) = (-1)^{i+j}M(i,j) = \mathring{A}_{ij}$, the formula (2.2) is

$$|A| = \sum_{j=1}^{n} a_{ij}\cdot\mathring{A}_{ij} = \sum_{i=1}^{n} a_{ij}\cdot\mathring{A}_{ij}. \quad (2.3)$$

This is the famous Laplace expansion [10].

In formula (2.2), the sum contains $\binom{n}{k}$ terms, when n is big, the calculation is a heavy work. But for tridiagonal matrix, there are only two nonzero terms, we can use formula (2.2) to evaluate the determinant $|H(n-1)|$ or $|D_{n-1}|$ to proof the Fibonacci identities (1.7).

Multiplying the entries 1 and i (or -1 in $|D_{n-1}|$) in the first row by the corresponding cofactors $\mathring{A}_{11} = |H(n-2)| = F_{n-1}$ and $\mathring{A}_{12} = (-1)^{1+2}\cdot i\cdot|H(n-3)| = F_{n-2}$, using the Laplace expansion (2.3), we obtain

$$|H(n-1)| = |H(n-2)| + (-1)^{1+2}\cdot i\cdot i\cdot|H(n-3)| = F_{n-1} + F_{n-2}. \quad (2.4)$$

Using the initial value $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, we can rewrite (2.4) as

$$F_n = F_{n-1} + F_{n-2} = F_2F_{n-1} + F_1F_{n-2}. \quad (2.5)$$

When we choose the first two rows of $H(n-1)$, there are only three 2×2 submatrices of $H(n-1)$ which determinants are not nonzero:

$A([1,2],[1,2]) = H(2) = F_3, A([1,2],[1,3]) = \begin{vmatrix} 1 & 0 \\ i & i \end{vmatrix} = i, A([1,2],[2,3]) = \begin{vmatrix} i & 0 \\ 1 & i \end{vmatrix} = -1$. Their corresponding cofactors are: