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An algorithm for low-rank approximation of bivariate functions using splines

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ABSTRACT

We present an algorithm for the approximation of bivariate functions by "low-rank splines", that is, sums of outer products of univariate splines. Our approach is motivated by the Adaptive Cross Approximation (ACA) algorithm for low-rank matrix approximation as well as the use of low-rank function approximation in the recent extension of the `chebfun` package to two dimensions. The resulting approximants lie in tensor product spline spaces, but typically require the storage of far fewer coefficients than tensor product interpolants. We analyze the complexity and show that our proposed algorithm can be efficiently implemented in terms of the cross approximation algorithm for matrices using either full or row pivoting.

We present several numerical examples which show that the performance of the algorithm is reasonably close to the best low-rank approximation using truncated singular value decomposition and leads to dramatic savings compared to full tensor product spline interpolation.

The presented algorithm has interesting applications in isogeometric analysis as a data compression scheme, as an efficient representation format for geometries, and in view of possible solution methods which operate on tensor approximations.

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1. Introduction

In recent years, low-rank and tensor approximation methods have increasingly found applications to many problems in numerical analysis and scientific computing. In boundary element methods (BEM), it was found that the use of so-called $\mathcal{H}$-matrices for the data-sparse representation of otherwise dense BEM matrices allows for the quasi-optimal realization of the BEM in three dimensions. The core idea is the hierarchical decomposition of the computational domain and the approximation of the resulting submatrices in low-rank form by the so-called Adaptive Cross Approximation (ACA) algorithm. These approximations have been extensively studied, and we can cite here only a few publications by Hackbusch [1], Hackbusch and Khoromskij [2], and Bebendorf [3]. The idea of low-rank approximation has been extended to tensors of orders higher than two, and the resulting tensor approximation schemes have been applied very successfully to many computational problems. We refer to the survey [4] and the monograph [5] as well as the references therein for an overview of this rapidly expanding field.

Isogeometric analysis (IGA) is a discretization method for partial differential equations introduced in [6] based on the idea that geometry representations from industry-standard CAD systems in terms of tensor product spline spaces should

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be used directly in analysis, and also solution fields should be represented in such spline spaces. This approach has become increasingly popular in the last decade, and tensor product B-spline and NURBS spaces still seem to be the most popular choices as approximation spaces in the IGA literature.

Therefore, one motivation of the present work is to study to what extent low-rank approximation methods can yield gains in computational efficiency in isogeometric analysis. As a first step, we study here the approximation of functions using low-rank approximations, which is important both to give an idea of the savings that can be expected and as a prerequisite for the future development of PDE solvers operating on tensor approximations directly. We restrict ourselves to the two-dimensional case, where we do not have to use the more sophisticated models for high-dimensional tensor approximation and can instead rely on simple sums of rank 1 matrices.

Some related work is given by Townsend and Trefethen [7] who describe an extension of the `chebfun` software to two dimensions. `chebfun` is a Matlab package for numerical computation with functions based on polynomial interpolation in Chebyshev nodes. The two-dimensional extension, `chebfun2`, is based on low-rank approximation of bivariate functions and representing the univariate factors by the techniques developed in `chebfun`. Their work also contains an overview of related techniques for low-rank approximation of functions; see [7, Section 2.3].

We base our algorithm on similar ideas as `chebfun2`, however we use splines instead of polynomial interpolation as the underlying univariate representation. Furthermore, we show that the use of a fixed tensor grid leads to advantages in the computational realization of the algorithm. We also give an algorithm using row pivoting rather than the full pivoting used in [7], which leads to dramatic time savings in large-scale approximation problems with only modestly increased error.

The remainder of the paper is structured as follows.

In Section 2, we collect some preliminaries on low-rank approximation of bivariate functions, in particular known results on best approximation by the truncated singular value decomposition, as well as on B-splines. We also state the cross approximation algorithm for low-rank approximation of matrices.

In Section 3, we develop our algorithm for low-rank approximation of bivariate functions. We base our algorithm on the translation of the ACA algorithm to functions and then introduce a discrete version of this algorithm by means of spline interpolation. We show how the resulting method can be efficiently realized in terms of the matrix ACA algorithm and a postprocessing step by spline interpolation. We give an analysis of the computational complexity.

In Section 4, we present several numerical examples. In particular, we study how close the results from the cross approximation algorithm are to the best possible approximation by truncated singular value decomposition, and we compare the errors obtained from full pivoting and the more efficient row pivoting. We also make some observations on how the choice of the underlying spline space influences the approximation quality of the algorithm.

In Section 5, we summarize the obtained results and discuss possible applications and further developments. We also compare our approach to prior work.

2. Preliminaries

2.1. Low-rank approximation of bivariate functions and the singular value decomposition

In this section, we summarize some results on the low-rank approximation of bivariate functions and in particular the best approximation result by the truncated singular value decomposition of a compact operator. In the matrix case, the fact that the best low-rank approximation can be obtained from the singular value decomposition is a classical result due to Eckart and Young [8]. The results for bivariate functions given here are the straightforward generalization of the matrix case using the spectral theory of compact operators on Hilbert spaces.

Let

$$\Omega = (-1, 1)^2, \qquad f : \Omega \to \mathbb{R}.$$

Our aim is to find univariate functions and coefficients

$$u_k, v_k \in L_2(-1, 1) \quad \text{and} \quad \sigma_k \in \mathbb{R} \quad \forall k \in \{1, \dots, K\}$$

such that f is close in some sense to the *low-rank approximation*

$$f_K(x, y) = \sum_{k=1}^{K} \sigma_k (u_k \otimes v_k)(x, y) := \sum_{k=1}^{K} \sigma_k u_k(x) v_k(y). \tag{1}$$

The best possible approximation for a given rank K is given by the truncated singular value decomposition (SVD). Assuming that $f \in L_2(\Omega)$, we introduce the integral operator

$$\begin{aligned} \mathcal{S} : L_2(-1, 1) &\to L_2(-1, 1) \\ u &\mapsto \int_{-1}^{1} f(x, \cdot) u(x)\, dx. \end{aligned}$$

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