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Identification problems with given material interfaces

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ABSTRACT

This paper is devoted to the identification of coefficients in scalar elliptic partial differential equations using an optimal control approach. The coefficients usually can be interpreted as material characteristics and play the role of the control variable. The paper focuses on processes in domains which can be split into a finite number of materially homogeneous subdomains, i.e. the coefficients to be identified are piecewise constant on them. In addition, we assume that the material interfaces are a priori known. We prove the existence of at least one solution of the optimal control problem for a large class of cost functionals and show that solutions can be obtained as limits of solutions to the problems which are governed by finite element discretizations of the state equations. Further, the unified algebraic sensitivity analysis of the first and the second order for several least squares type cost functions is investigated. Finally, a model problem of the identification of coefficients characterizing hydraulic conductivity by pumping tests in groundwater flow modeling is numerically solved using the sequential approach and the scalarization technique.
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1. Introduction

Many processes which are important in engineering, health care, science, etc. are described by partial differential equations (PDEs). Mathematical modeling for predicting the behavior of systems requires the selection of proper control variables and equations describing the process. Control variables may have different nature. This paper deals with the identification of coefficients in the 2nd order scalar elliptic equations. The coefficients of PDEs mostly represent material properties of a structure and their proper determination can be based on the knowledge of the state problem solution for a given input data. To this end we use an optimal control approach whose abstract form reads as follows:

$$\text{find } a^* \in \underset{a\in\mathcal{U}_{ad}}{\operatorname{argmin}}\, J(a, u(a)) \tag{1.1}$$

where J is a cost functional depending on: control variables a (coefficient of PDE), that are required to be elements of a set of admissible controls $\mathcal{U}_{ad}$ and the solution $u(a)$ to the state problem corresponding to $a \in \mathcal{U}_{ad}$. A particular choice of J depends on the goals we want to achieve. One of the most frequently used functionals in practical applications is a least squares type

$$J\,(a, u(a)) = \frac{1}{2}\,\|R\,u(a) - z_d\|^2\,. \tag{1.2}$$

Here R stands for the selection (observation) operator, z_d is a target and $\|\bullet\|$ is an appropriate norm. The straightforward approach to the identification problem (1.1) combines a suitable minimization method with a procedure for computing

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the solution to the state problem. If a gradient type minimization method is used then an extra procedure for the efficient computation of derivatives of the objective functional has to be involved. In many applications (and this is just our case) the set $\mathcal{U}_{ad}$ is (isomorphic to) a compact subset of the Euclidean space $\mathbb{R}^r$.

This fact simplifies a lot the mathematical analysis since no additional constraints ensuring compactness of $\mathcal{U}_{ad}$ are needed. The simple case is introduced in Section 2: the coefficients of the PDEs are piecewise constant in a fixed number of subdomains $\{\Omega_i\}_{i=1}^r$, that define a partition of a domain of definition Ω of the state problem with a priori known interfaces. Such situations arise in many applications when the decomposition of Ω into materially homogeneous sub-domains Ω_i can be detected either by various types of tomography (e.g. [1]) or less rigorously just by an engineering guess (e.g. [2]). The obtained results remain valid when besides the (piecewise constant) coefficients in $\{\Omega_i\}_{i=1}^r$ also interfaces among them are the object of the identification, provided that the shapes of all Ω_i are described by a finite number of parameters. The existence of a solution to (1.1) is based on compactness and continuity arguments. To prove continuity of the composed functional $\mathcal{J}(a) := J(a, u(a))$, Section 2 analyzes continuity of the control-to-state mapping $\varphi : \mathcal{U}_{ad} \mapsto H^1(\Omega)$, $\varphi(a) = u(a)$ and also continuity of $\tilde{\varphi} : \mathcal{U}_{ad} \mapsto H^2(\Omega')$, $\tilde{\varphi}(a) = u(a)_{|\Omega'}$, where Ω' is a sub-domain lying strictly inside of Ω_i for some i. The latter result is important in the case when J uses pointwise targets. Moreover we proved that any accumulation point of the sequence of solutions to the discretized identification problems when the discretization parameter tends to zero is a solution to the original continuous setting. Section 3 deals with computing the first and second derivatives of the discretized least squares type cost function of the form

$$\mathcal{J}(\boldsymbol{a}) = \frac{1}{2}\left(\boldsymbol{B}(\boldsymbol{a})\left(\boldsymbol{R}\boldsymbol{u}(\boldsymbol{a}) - \boldsymbol{z_d}\right), \boldsymbol{R}\boldsymbol{u}(\boldsymbol{a}) - \boldsymbol{z_d}\right)_s, \tag{1.3}$$

where $\boldsymbol{a}$ is the vector representation of $a \in \mathcal{U}_{ad}$, $\boldsymbol{B}(\boldsymbol{a})$ is a symmetric, positive definite matrix, $\boldsymbol{R}$ is the matrix representation of the observation mapping R, $\boldsymbol{z_d}$ is a target vector, $\boldsymbol{u}(\boldsymbol{a})$ is the solution of a linear algebraic system $\boldsymbol{K}(\boldsymbol{a})\boldsymbol{u}(\boldsymbol{a}) = \boldsymbol{b}$, and $(\bullet, \bullet)_s$ stands for the Euclidean inner product in $\mathbb{R}^s$. This form of $\mathcal{J}$ is quite general. According to the choice of the matrices $\boldsymbol{B}(\boldsymbol{a})$ and $\boldsymbol{R}$ one gets the Euclidean, energy and equation error functions, respectively. The expressions for the corresponding gradient, Hessian as well as the Gauss–Newton approximations to the Hessian are derived. The use of the Euclidean error functional is standard, see e.g. [3], the use of the energy and equation error function was inspired by [4–6] and the references therein. One of the advantages of the last two mentioned error functions is the fact that in the case of targets distributed in the whole Ω one can easily derive the expression for the full Hessian $\mathbb{D}^2\mathcal{J}$ and show that $\mathbb{D}^2\mathcal{J}$ is positive semi-definite, i.e. the minimized functions are convex. Section 4 is devoted to the optimization techniques which are used in the last section. First, we recall the Levenberg–Marquardt method frequently used in identification problems. Due to the demanding character of measurements the only way how to get more data is to repeat them for different input data (multi-response). This leads to an identification problem involving more cost functions. For solving such a type of problems two approaches are proposed: a sequential method and the weighted sum method. In Section 5 the identification problem for coefficients characterizing hydraulic conductivity by using pumping tests is formulated. This model example is a modification of the one in [7]. It has a structure which is strongly different from the standard setting introduced in Section 2. The characteristic features of this example are the following: (i) only pointwise targets are available and their number is very small (only eight in our case), (ii) conductivities are piecewise constant. These are identified only in a small domain $\omega \subset \Omega$ on the boundary of which all measurement points are located, (iii) the right hand side of the state problem is given by two Dirac distributions, (iv) the multi-response character of gaining data: eight measurement points are split into four disjoint pairs. With any such pair one experiment which provides the input data in the remaining 6 points is associated. In this way four least squares type functions are generated.

Finally, in Section 6 we use the sequential and weighted sum method with the Euclidean error function for solving the model example considering exact and noisy data.

The paper addresses inverse identification problems characterized by piecewise constant (material) coefficients, some a priori knowledge of domains, where coefficients can be supposed as constant, mostly pointwise (not distributed) measurements and utilization of multi-response data. Such problems frequently appear in engineering applications, see e.g. [8].

The contribution of the presented paper can be seen in

- simple and straightforward existence and convergence analysis for a class of identification problems which also involves cost functions defined by pointwise measurements;
- unified sensitivity analysis: computations of the gradients and Hessians or Gauss–Newton approximations to the Hessians for a large class of least squares type functions including Euclidean, energy and equation error cost functions;
- efficient numerical realization of a model multi-response problem taken from hydrogeology.

2. Identification problems with given material interfaces—the scalar case

The aim of this section is to give the mathematical justification of problems arising in identification of materials separated by interfaces whose position is known. To this end we use an optimal control approach in which coefficients of partial differential equations play the role of control variables. We restrict ourselves to the identification of coefficients in a *scalar*

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