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Discretization error estimates in maximum norm for convergent splittings of matrices with a monotone preconditioning part

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ABSTRACT

For finite difference matrices that are monotone, a discretization error estimate in maximum norm follows from the truncation errors of the discretization. It enables also discretization error estimates for derivatives of the solution. These results are extended to convergent operator splittings of the difference matrix where the major, preconditioning part is monotone but the whole operator is not necessarily monotone.

1. Introduction

Finite difference methods for elliptic problems [1–3] are most suitable for regular grids. They can often be formulated so that the discrete operator is a monotone matrix. This enables a simple estimate of the discretization error in maximum norm with a best constant factor in the upper bound. In addition, one can approximate first and higher order derivatives of the solution with the same order of convergence, if the solution is sufficiently regular. The estimates are local, and therefore hold even for problems with discontinuous coefficients on macroelements, if the solution is regular in the interior of each element. One can consider here both rectangular and hexagonal meshes. The difference operator is particularly simple for hexagonal meshes. High order difference approximations can be constructed either with use of approximations on locally extended meshes of the higher order derivative terms in the truncation error, or with use of extrapolation for regularly refined meshes.

In the case when the matrix is not monotone, one can split it in a monotone and remainder term. If this, with proper scaling of the matrices, leads to a convergent splitting, one can still estimate the maximum norm of the error, but with a factor that becomes larger when the splitting leads to a larger convergence factor. This approach can be illustrated for the Helmholtz equation. Other possible applications might arise for the system of elasticity equation, using a splitting into the divergence and grad div terms, or proper splittings can be based on equivalent operator pairs, see [4] for examples of equivalent operators.

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In this paper, first discretization error estimates for monotone matrices are described. In Section 3 some high order difference approximations are derived. Estimates using convergent splittings are presented in Section 4, which includes the case of the Helmholtz equation, illustrated by a numerical test.

2. Preliminaries: discretization error estimates for monotone matrices

Recall that a discretization matrix $\mathcal{L}_h$ on a difference mesh Ω_h, where h denotes the mesh size, is called *monotone* if

$$\mathcal{L}_h u \geq 0 \quad \text{implies } u \geq 0. \tag{2.1}$$

It is readily seen that monotone matrices are nonsingular, because if $\mathcal{L}_h u \leq 0$ then $\mathcal{L}_h(-u) \geq 0$ so $-u \geq 0$, i.e. $u \leq 0$. Hence, if $\mathcal{L}_h u = 0$ then both $u \geq 0$ and $u \leq 0$, that is $u = 0$. Further, a nonsingular discretized operator (matrix) $\mathcal{L}_h = A$ is monotone if and only if $A^{-1} \geq 0$, i.e. the entries of its inverse are nonnegative. The sufficiency follows immediately. To show the necessity part, if A^{-1} contains a negative entry in position (i, j), then $A_h u = e_j$ (the jth unit vector) has a solution with ith component $u_i = (A_h^{-1} e_j)_i < 0$, so A_h cannot be monotone [5].

If $A = M - R$, where M is monotone and $M^{-1}R \geq 0$, and the splitting is convergent, i.e. $\varrho(M^{-1}R) < 1$, where $\varrho(\cdot)$ is the spectral radius, then A is monotone. Such a splitting is called a convergent weak regular splitting [6]. This is seen simply by expanding the inverse of $M^{-1}A = I - M^{-1}R$ in a Neumann series. In many applications one gets a difference operator $\mathcal{L}_h = D - R$ of positive type, i.e. where D is monotone, and $R \geq 0$. Then this operator is monotone if $\varrho(D^{-1}R) < 1$. If D is diagonal, such a matrix is called a diagonally dominant M-matrix [7].

A major advantage of dealing with monotone matrices is that the inverse of the matrices is bounded in maximum norm, $\|\cdot\|_\infty$, uniformly with respect to the mesh parameter h. This leads to a simple and useful discretization error estimate. To see this, let $v \geq 0$ be a normalized vector, i.e. $\max_i v(x_i) = 1$ for which $\mathcal{L}_h v \geq \alpha$, $\alpha > 0$. (Such a vector or function is called a barrier function for the operator.) Then with $\mathcal{L}_h v = \alpha e$,

$$1 = \|v\|_\infty = \alpha \|\mathcal{L}_h^{-1} e\|_\infty,$$

where $e^T = (1, 1, \ldots, 1)$. Hence

$$\|\mathcal{L}_h^{-1}\|_\infty = \|\mathcal{L}_h^{-1} e\|_\infty \leq \frac{1}{\alpha}.$$

Therefore the best constant, $\|\mathcal{L}_h^{-1}\|_\infty$ can be computed by solving $\mathcal{L}_h v = e$. Now let $\mathcal{L}_h v_h = f_h$ be the discrete equation for an elliptic differential operator $\mathcal{L}u = f$, where $\mathcal{L}_h$ is monotone. Then

$$\mathcal{L}_h(u - u_h) = \mathcal{L}_h u - f_h \tag{2.2}$$

is the truncation error, and the discretization error can be estimated by

$$\|u - u_h\|_\infty \leq \frac{1}{\alpha} \|\mathcal{L}_h u - f_h\|_\infty .$$

For regular problems, i.e. with a sufficiently differentiable solution u, one can estimate the truncation error

$$\tau_h := \mathcal{L}_h u - f_h$$

from a Taylor series expansion. Note that the remainder term of $O(h^k)$ in the Taylor expansion can be written in integral form as $\int_x^{x+h} (x + h - s)^{k-1}/(k-1)!\, u^{(k)}(s) ds$.

There are various ways one can further improve the accuracy of the discrete solution. One can estimate the lowest order derivative terms in the Taylor expansion by use of difference approximations on a locally extended mesh or one can use higher order difference approximations, see Section 3. Another way is by extrapolating the solution on a mesh and its refinement. To show this, let $\mathcal{L}u = f$ be a second order elliptic differential operator approximated by a difference operator with second order truncation error. Assume for simplicity given Dirichlet boundary conditions and assume that the solution $u \in C^6(\Omega)$. Let the truncation error satisfy

$$\mathcal{L}_h(u - u_h) = \mathcal{L}_h u - f = h^2 G u + O(h^4),$$

where G is a differential operator of fourth order. As an example, let $\mathcal{L} = -\triangle$ be the Laplacian, then for a rectangular mesh $Gu = -\frac{1}{12}(u_x^{(4)} + u_y^{(4)})$. Further let φ be the solution of

$$\mathcal{L}\varphi = Gu \quad \text{in } \Omega, \ \varphi = 0 \text{ on } \partial\Omega.$$